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Simplex stochastic collocation with ENO-type stencil selection for robust uncertainty quantification

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ABSTRACT

Multi-element uncertainty quantification approaches can robustly resolve the high sensitivities caused by discontinuities in parametric space by reducing the polynomial degree locally to a piecewise linear approximation. It is important to extend the higher degree interpolation in the smooth regions up to a thin layer of linear elements that contain the discontinuity to maintain a highly accurate solution. This is achieved here by introducing Essentially Non-Oscillatory (ENO) type stencil selection into the Simplex Stochastic Collocation (SSC) method. For each simplex in the discretization of the parametric space, the stencil with the highest polynomial degree is selected from the set of candidate stencils to construct the local response surface approximation. The application of the resulting SSC–ENO method to a discontinuous test function shows a sharper resolution of the jumps and a higher order approximation of the percentiles near the singularity. SSC–ENO is also applied to a chemical model problem and a shock tube problem to study the impact of uncertainty both on the formation of discontinuities in time and on the location of discontinuities in space.

1. Introduction

Resolving discontinuities in stochastic problems is important, since they can lead to high sensitivities with respect to input uncertainties. They can also result in oscillatory approximations and, consequently, in the prediction of non-zero probabilities for unphysical realizations such as negative static pressures. In order to avoid these problems, the polynomial interpolation degree can locally be reduced to a linear approximation to avoid overshoots at the discontinuity in a multi-element uncertainty quantification (UQ) approach. In this context, two points are essential to maintain a highly accurate solution despite the locally first degree approximation. Firstly, the region in which the interpolation is reduced to a piecewise linear function should be as small as possible. This means that the samples need to be concentrated around the discontinuity to pinpoint its location. Secondly, the higher degree interpolation in the smooth regions should be extended as close as possible up to the discontinuity to maintain high order accuracy near the singularity. These two objectives are achieved here by introducing an Essentially Non-Oscillatory (ENO) type stencil selection into the Simplex Stochastic Collocation (SSC) method.

The ENO scheme has been developed by Harten and Osher [13] as a robust spatial discretization in the finite volume method (FVM) for deterministic Computational Fluid Dynamics (CFD) [14]. In that field, the robust approximation of discontinuities is critical for resolving shock waves and contact surfaces in the flow field. Therefore, it was proposed by Abgrall [1] and Barth [6] to use shock-capturing FVM to discretize also the parametric space to obtain robust approximations for

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stochastic CFD. These FVM discretizations of the combined physical and parametric space use the ENO and Weighted Essentially Non-Oscillatory (WENO) [16] schemes. Since FVM has originally been developed for the three-dimensional physical space, the direct extension of FVM to these high dimensional parametric spaces can, however, be inefficient due to the curse-of-dimensionality. Because of the different nature of the parametric space, there are also no physical fluxes between the cells in the stochastic directions, which form the basis of FVM. Therefore, we follow a different approach to extend the robustness of FVM to parametric space. We reformulate the robustness *principles* of FVM in terms of the parametric space and develop new UQ methods that satisfy these concepts in the stochastic dimensions. This approach has the advantages that it maintains the FVM robustness in combination with the effectiveness of specifically designed methods for UQ, which can also be used with other spatial discretizations than FVM. In this way, we have previously introduced, for instance, the Total Variation Diminishing (TVD) [12,31], Extremum Diminishing (ED) [15,33], and Local Extremum Diminishing (LED) [15,35] principles into UQ and proposed the Essentially Extremum Diminishing (EED) [34] concept.

The ENO spatial discretization [13] achieves an essentially non-oscillatory approximation of the solution of hyperbolic conservation laws. Non-oscillatory means, in this context, that the number of local extrema in the solution does not increase with time. The ENO scheme obtains this property using an adaptive-stencil approach with a uniform polynomial degree for reconstructing the spatial fluxes. Each spatial cell X_j is assigned r stencils $\{S_{j,i}\}_{i=1}^{r}$ of degree p, all of which include the cell X_j itself. Out of this set of candidate stencils $\{S_{j,i}\}$, the stencil S_j is selected for cell X_j that results in the interpolation $w_j(\mathbf{x})$ which is smoothest in some sense based on an indicator of smoothness $\mathrm{IS}_{j,i}$. In this way, a cell next to a discontinuity is adaptively given a stencil consisting of the smooth part of the solution, which avoids Gibbs-like oscillations in physical space. Attention has been paid to the efficient implementation of ENO schemes by Shu and Osher [24,25]. Fig. 1 shows an example of the ENO stencil selection in a FVM discretization of a discontinuity in one spatial dimension using piecewise quadratic polynomials.

ENO-type stencil selection is here used in the SSC multi-element UQ method to obtain an accurate approximation of discontinuities in parametric space. Multi-element UQ methods discretize the stochastic dimensions using multiple subdomains comparable to spatial discretizations in physical space. These local methods [3,19,28] can be based on Stochastic Galerkin (SG) projections of Polynomial Chaos (PC) expansions [10,36] in each of the subdomains. Other methods [2,8,17] use a Stochastic Collocation (SC) approach [4,37] to construct the local polynomial approximations based on sampling at quadrature points in the elements. These methods commonly use sparse grids of Gauss quadrature rules in hypercube subdomains combined with solution-based refinement measures for resolving nonlinearities. Because of the hypercube elements, these methods are most effective in capturing discontinuities that are aligned with one of the stochastic coordinates.

In contrast, the SSC method [35,34] is based on a simplex tessellation of the parametric space with sampling points at the vertexes of the simplex elements. The polynomial approximation in the simplexes Ξ_j is built using higher degree interpolation stencils S_j, with local polynomial degree p_j, consisting of samples in the vertexes of surrounding simplexes. The degree p_j is controlled by a Local Extremum Conserving (LEC) limiter, which reduces p_j and the stencil size to avoid overshoots in the interpolation of the samples where necessary. The limiter, therefore, leads to a non-uniform polynomial degree that reduces to a linear interpolation in simplexes which contain a discontinuity and that increases away from singularities. SSC employs adaptive refinement measures based on the hierarchical surplus and the geometrical properties of the simplexes to identify the location of discontinuities. However, the limiter can result in an excessive reduction of the polynomial degree also at significant distances away from a discontinuity. Since the polynomial degree affects the refinement criteria, this can also deteriorate the effectiveness of the refinement to sharply resolve singularities.

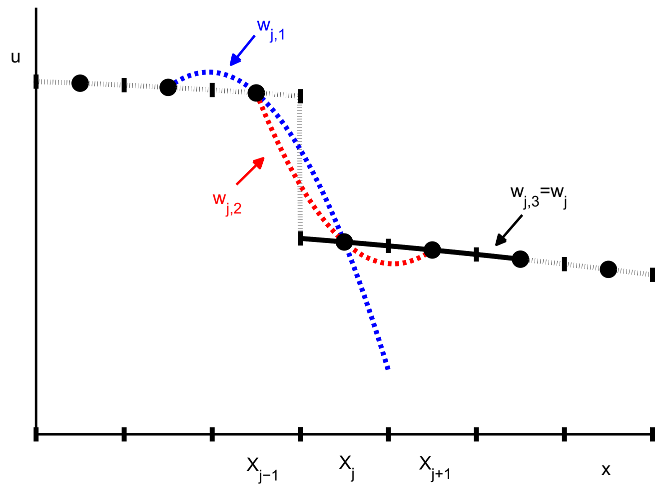

Fig. 1. ENO stencil selection for the quadratic reconstruction $w_j(x)$ in the spatial cell X_j out of the candidates $\{w_{j,1}, w_{j,2}, w_{j,3}\}$ for cell-centered FVM in one physical dimension.