Journal of Computational Physics 294 (2015) 1–19

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Mercer's spectral decomposition for the characterization of thermal parameters

E. Ahusborde [a], M. Azaïez [b,*], F. Ben Belgacem [c], E. Palomo Del Barrio [d]

[a] LMA, UMR CNRS 5142, Université de Pau et des Pays de l'Adour, France
[b] Université de Bordeaux, IPB, I2M, CNRS, UMR 5295, 33607 Pessac, France
[c] Sorbonne Universités, Université de Technologie de Compiègne, LMAC, EA 2222, F-60205 Compiègne, France
[d] Université de Bordeaux, I2M, CNRS UMR 5295, 33405 Talence, France

ARTICLE INFO

Article history:
Received 14 April 2014
Received in revised form 5 March 2015
Accepted 7 March 2015
Available online 30 March 2015

Keywords:
Singular value decomposition
Spectral expansion of Mercer kernels
Identification

ABSTRACT

We investigate a tractable Singular Value Decomposition (SVD) method used in thermography for the characterization of thermal parameters. The inverse problem to solve is based on the model of transient heat transfer. The most significant advantage is the transformation of the *dynamic* identification problem into a *steady* identification equation. The time dependence is accounted for by the SVD in a performing way. We lay down a mathematical foundation well fitted to this approach, which relies on the spectral expansion of Mercer kernels. This enables us to shed more light on most of its important features. Given its potentialities, the analysis we propose here might help users understanding the way the SVD algorithm, or the TSVD, its truncated version, operate in the thermal parameters estimation and why it is relevant and attractive. When useful, the study is complemented by some analytical and numerical illustrations realized within MATLAB's code.

1. Introduction and the heat equation

Advances in technologies of infrared cameras have considerably increased the data acquisition frequency and tremendously improved the lateral resolution. Infrared thermography, as a predictive tool for the characterization of thermal properties of material, is therefore positively impacted and highly enhanced. The counterpart is that the amount of noisy data to be treated, in inverse formulations which can be strongly unstable, has exploded. Common and crude approaches for estimating thermal diffusivities and heat transfer coefficients become computationally expensive and their use may be hence questionable because of the size of the least squares minimization problem to solve (see [14,17]). Reducing computing efforts can possibly be achieved by a point-by-point least squares estimation as in [22,5]. This technique is very sensitive to noise because one should handle space and especially *time derivatives* of the temperature data. An affordable way to control the noise effect on the final results consists in fixing particular experimental conditions so that the temperature field, solution of the transient heat conduction problem, has an accessible closed form (see [20] and references therein). Usually, this requires exciting the sample by a point-like or a line-like intensity modulated laser beam, selecting a suitable frequency for laser modulation. Efficient simplifications of the characterization process implies thus a reducing flexibility and an increasing of the complexity in the experimentation.

* Corresponding author.
E-mail address: azaiez@enscbp.fr (M. Azaïez).

http://dx.doi.org/10.1016/j.jcp.2015.03.037
0021-9991/

The method we are particularly interested in has been used in the works by Karhunen [18] and Loève [19] to provide useful finite or infinite expansions of stochastic processes on well fitted orthogonal basis. Dependence on the time and the random variables are therefore separated. This uncoupling is enabled by Mercer's decomposition theorem and has a wide impact in many theoretical and practical areas such as the theory of signals for instance. In numerous applications users actually consider the truncated Karhunen–Loève expansions where most relevant features of signals to study are represented by few modes. It is particularly popularized in fluid mechanics, by the work in [4]. This basic idea can also be applied to our inverse problem (see [12]). The main advantage we see is the efficient way the time dependency is accounted for. This step may be viewed as if the singular value decomposition is used for transforming the inverse problem of thermal parameters estimation in transient heat conduction to a one related to a steady heat equation. In the reduced problem, the available observation turns out to be the whole stiffness matrix of the steady equation, calculated on the basis provided by Mercer's theorem. The reduced problem may be considered in a full or a partial form and then solved by a least squares procedure consolidated by some regularization strategies. This methodology allows to deal efficiently with a large amount of noisy data without requiring analytical solutions, which enlarge the form of thermal loads engineers can apply to the sample. We greatly recommend the papers [12,13] for an argumented illustration of the efficiency of the approach. Our focus here is on a mathematical discussion, complemented with some analytical calculations and numerical examples, so as to extract the major properties of this method which help us to understand its foundation and robustness.

Before completely describing the problematic of reconstruction thermal coefficient, let us consider the direct transient heat transfer equation. Assume Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with a regular boundary $\partial\Omega$. Let t_F be a positive real number. We define the space–time domains

$$Q =]0, t_F[\times \Omega, \qquad \Sigma =]0, t_F[\times \partial\Omega.$$

The generic point in Ω is denoted by $\boldsymbol{x}$ and $\boldsymbol{n}$ stands for the unit normal vector to $\partial\Omega$ which is outward to Ω. The heat model we deal with consists in: *finding a temperature field T such that*

$$\begin{aligned}
&\frac{\partial T}{\partial t} - \operatorname{div}(D\nabla T) + \beta T = 0, \qquad \text{in } Q,\\
&(D\nabla T)\cdot\boldsymbol{n} = 0, \qquad \text{on } \partial\Omega,\\
&T(0,\cdot) = T_0, \qquad \text{in } \Omega.
\end{aligned}$$

The diffusion is anisotropic here. D is hence a symmetric and positive definite second order tensor while β is the heat transfer coefficient. These parameters are actually rescaled by the thermal capacity (ρC_p) which means that

$$D := \frac{D}{(\rho C_p)}, \qquad \beta := \frac{\beta}{(\rho C_p)}.$$

Applying the variational principle yields to the following formulation: *find $T \in L^2(0, t_F; H^1(\Omega))$ satisfying $T(0,\cdot) = T_0$ and such that*

$$\frac{\partial}{\partial t}\int_\Omega T\, T'\, d\boldsymbol{x} + \int_\Omega D\nabla T\cdot\nabla T'\, d\boldsymbol{x} + \int_\Omega \beta T\, T'\, d\boldsymbol{x} = 0, \quad \forall T' \in H^1(\Omega). \tag{1}$$

For an initial condition T_0 taken in $L^2(\Omega)$, the solution T lies in $\mathscr{C}([0,t_F], L^2(\Omega)) \cap L^2(0,t_F; H^1(\Omega))$ and the following stability holds (see [8,9]),

$$\|T(t_F)\|^2_{L^2(\Omega)} + \int_{(0,t_F)}\int_\Omega D\nabla T\cdot\nabla T\, d\boldsymbol{x}dt + \int_{(0,t_F)}\int_\Omega \beta T^2\, d\boldsymbol{x}dt \leq \|T_0\|^2_{L^2(\Omega)}.$$

The problem we investigate is inverse. It is concerned with the identification of the thermal parameters, the conductivity D and the heat transfer coefficient β, from the knowledge of the temperature field. Notice that the mapping, $(D,\beta) \mapsto T$, is *non-linear* and its inversion is numerically unstable. Problems of parameter identification for the heat equation are well known to be hard to deal with. We refer to [1,6] for some discussions of such issues.

The real problem we are interested in may be described as follows. The sample to be tested is a thin plate, surrounded by a uniform media. The initial state T_0 is generated by a thermal excitation of the sample by a laser beam. The applied heat flux is switched off at an instant taken as the initial time ($t = 0$). Then, the thermal relaxation of the material is considered and the temperature field is measured. The time/space resolution is assumed sufficiently high to expect a rich approximation of the temperature field. Using infrared camera enables users to collect a great amount of records. Infrared cameras allows for rich representation of the temperature field. The issue consists afterwards in recovering D and β from the knowledge of the temperature field T. The technical tool we use is the SVD method, based on the Mercer expansion theorem. To describe it we assume that T is entirely available. The first step to carry out is the transformation of the unsteady inverse problem into a parameters identification for a steady problem. The methodology we focus on here is the one already followed by many authors, in particular in [12,13] to solve the identification problem of the diffusion and heat transfer coefficients. Proposing a theoretical analysis of some issues related to the method is the main novelty here,