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A quasi-optimal domain decomposition algorithm for the time-harmonic Maxwell's equations

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ABSTRACT

This paper presents a new non-overlapping domain decomposition method for the time harmonic Maxwell's equations, whose effective convergence is quasi-optimal. These improved properties result from a combination of an appropriate choice of transmission conditions and a suitable approximation of the Magnetic-to-Electric operator. A convergence theorem of the algorithm is established and numerical results validating the new approach are presented.

1. Introduction

In terms of computational methods, solving three-dimensional time-harmonic electromagnetic wave problems is known to be a challenging topic, especially in the high frequency regime. Among the various approaches that can be used to solve such problems, the Finite Element Method (FEM) with an Absorbing Boundary Condition (ABC) or a Perfectly Matched Layer (PML) is widely used for its ability to handle complex geometrical configurations and materials with non-homogeneous electromagnetic properties [20]. However, the brute-force application of the FEM in the high-frequency regime leads to the solution of very large, complex and possibly indefinite linear systems. Direct sparse solvers do not scale well for such problems, and Krylov subspace iterative solvers can exhibit slow convergence, or even diverge [16]. Domain decomposition methods provide an alternative, iterating between subproblems of smaller sizes, amenable to sparse direct solvers.

Improving the convergence properties of the iterative process constitutes the key in designing an effective algorithm, in particular in the high frequency regime. The optimal convergence is obtained by using as transmission condition on each interface between subdomains the so-called Magnetic-to-Electric (MtE) map [27] linking the magnetic and the electric surface currents on the interface. This however leads to a very expensive procedure in practice, as the MtE operator is non-local. A great variety of techniques based on *local* transmission conditions have therefore been proposed to build practical algorithms [10,1,12,15,29,28,30,11].

In the context of acoustic simulations, quasi-optimal local transmission conditions for domain decomposition methods were proposed in [7], based on high-order rational approximations of the Dirichlet-to-Neumann operator. In this paper,

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we analyze and validate the extension of this approach to electromagnetics, based on the high-order approximations of the MtE operator developed in [14]. We show that the resulting domain decomposition algorithm exhibits quasi-optimal convergence properties, i.e., that the convergence is optimal for the evanescent modes and significantly improved compared to competing approaches for the remaining modes. The finite element implementation of the method is available online as the open source software package GetDDM,[1] which combines the open source mesh generator Gmsh [19] and the finite element solver GetDP [13,18] for large scale domain decomposition simulations.

The paper is organized as follows. In Section 2 we introduce the scattering problem as well as the non-overlapping DDM. Optimized local transmission conditions are presented in Section 3. Section 4 develops a convergence analysis for this approximate transmission condition on a model problem. Section 5 details the complex Padé approximation of the square-root operator to get a local representation. Section 6 presents the finite element implementation of the resulting DDM. Numerical results on three-dimensional problems are presented in Section 7. The paper is concluded in Section 8 with perspectives for future work.

2. Problem setting and non-overlapping optimized Schwarz DDM

Let K be a bounded scatterer in $\mathbb{R}^3$ with smooth closed boundary Γ. The associated unbounded domain of propagation is denoted by $\Omega := \mathbb{R}^3 \backslash \overline{K}$. The exterior electromagnetic scattering problem by a perfectly conducting body K is given by

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - k^2\mathbf{E} = 0, & \text{in } \Omega, \\ \gamma^T(\mathbf{E}) = -\gamma^T(\mathbf{E}^{\text{inc}}), & \text{on } \Gamma, \\ \lim_{r\to\infty} r\left(\mathbf{E} - \frac{\imath}{k}\hat{\mathbf{x}} \times \mathbf{curl}\,\mathbf{E}\right) = 0. \end{cases} \tag{1}$$

In the above equations, $\mathbf{E}$ denotes the scattered electric field. The wavenumber is $k := 2\pi/\lambda$, where λ is the wavelength, and the unit imaginary number is $\imath = \sqrt{-1}$. The curl operator is defined by $\mathbf{curl}\,\mathbf{a} := \nabla \times \mathbf{a}$, for a complex-valued vector field $\mathbf{a} \in \mathbb{C}^3$. The nabla operator is $\nabla :=^t (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$, where $\mathbf{x} =^t (x_1, x_2, x_3) \in \mathbb{R}^3$. The notation $\mathbf{a} \times \mathbf{b}$ designates the cross product and $\mathbf{a} \cdot \bar{\mathbf{b}}$ the inner product between two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{C}^3$, where $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$. The associated norm is $||\mathbf{a}|| := \sqrt{\mathbf{a} \cdot \bar{\mathbf{a}}}$. Vector $\mathbf{n}$ is the unit outwardly directed normal to Ω and $\mathbf{E}^{\text{inc}}$ defines a given incident electric field. Let us consider a general domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$, $\mathbf{n}$ the outwardly directed unit vector to $\mathcal{D}$, then the tangential traces applications are defined by

$$\gamma^t : \mathbf{v} \mapsto \mathbf{v}_t := \mathbf{n} \times \mathbf{v}_{|\partial\mathcal{D}} \quad \text{and} \quad \gamma^T : \mathbf{v} \mapsto \mathbf{v}_T := \mathbf{n} \times (\mathbf{v}_{|\partial\mathcal{D}} \times \mathbf{n}).$$

Let us now write $\mathbf{x} = r\hat{\mathbf{x}} \in \mathbb{R}^3$, where $r := ||\mathbf{x}||$ is the radial distance to the origin and $\hat{\mathbf{x}}$ is the directional vector of the unit sphere $\mathbb{S}_1$. Then, the last equation of system (1), which is the so-called Silver–Müller radiation condition at infinity, provides the uniqueness of the solution to the scattering boundary-value problem (1).

To solve numerically (1) by a volume discretization method, it is standard to truncate the exterior domain of propagation by using a fictitious boundary Γ^∞ surrounding Ω. As a result, we have to solve the following problem in a bounded domain Ω, with boundaries Γ and Γ^∞,

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - k^2\mathbf{E} = 0, & \text{in } \Omega, \\ \gamma^T(\mathbf{E}) = -\gamma^T(\mathbf{E}^{\text{inc}}), & \text{on } \Gamma, \\ \mathcal{B}(\gamma^T(\mathbf{E})) - \frac{\imath}{k}\gamma^t(\mathbf{curl}\,\mathbf{E}) = 0, & \text{on } \Gamma^\infty. \end{cases} \tag{2}$$

The operator $\mathcal{B}$ can be exact, resulting then in a transparent boundary condition that avoids any spurious unphysical reflection. However, such a boundary condition is global since it is defined by a nonlocal boundary integral operator on Γ^∞ (i.e. the MtE operator Λ: $\gamma^T(\mathbf{E}) \mapsto \Lambda(\gamma^T(\mathbf{E})) = \gamma^t(\mathbf{curl}\,\mathbf{E})$). This generates a dense part in the global discretization matrix that must be solved at the end of the computational process. For reducing the cost of computation, a local Absorbing Boundary Condition (ABC) is generally preferred, which means that the operator $\mathcal{B}$ is in fact an approximation of Λ. Since the aim of this paper is not devoted to ABCs, we restrict ourselves to the simplest ABC: $\mathcal{B} = \mathbf{I}$ ($\mathbf{I}$ is the surface identity operator). This corresponds to the well-known Silver–Müller ABC at finite distance.

Let us now focus on the construction of optimized Schwarz Domain Decomposition Methods (DDM) without overlap [17,12,15,23,22,29,28,7,1,10,9,30,11] for the approximate boundary-value problem (2). The first step of the method [9,10] consists in splitting Ω into several subdomains Ω_i, $i = 1, \ldots, N_{\text{dom}}$, in such a way that

- $\overline{\Omega} = \bigcup_{i=1}^{N_{\text{dom}}} \overline{\Omega}_i$ $(i = 1, \ldots, N_{\text{dom}})$,
- $\Omega_i \cap \Omega_j = \emptyset$, if $i \neq j$ $(i, j = 1, \ldots, N_{\text{dom}})$,
- $\partial\Omega_i \cap \partial\Omega_j = \overline{\Sigma}_{ij} = \overline{\Sigma}_{ji}$ $(i, j = 1, \ldots, N_{\text{dom}})$ is the fictitious interface separating Ω_i and Ω_j as long as its interior Σ_{ij} is not empty.

[1] http://onelab.info/wiki/GetDDM.