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A robust moving mesh method for spectral collocation solutions of time-dependent partial differential equations

Christopher J. Subich *,1

University of Waterloo, 300 University Avenue West, Waterloo, Ontario, N2L 3G1, Canada

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ABSTRACT

This work extends the machinery of the moving mesh partial differential equation (MMPDE) method to the spectral collocation discretization of time-dependent partial differential equations. Unlike previous approaches which bootstrap the moving grid from a lower-order, finite-difference discretization, this work uses a consistent spectral collocation discretization for both the grid movement problem and the underlying, physical partial differential equation. Additionally, this work develops an error monitor function based on filtering in the spectral domain, which concentrates grid points in areas of locally poor resolution without relying on an assumption of locally steep gradients. This makes the MMPDE method more robust in the presence of rarefaction waves which feature rapid change in higher-order derivatives.

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1. Introduction and motivation

Spectral collocation (pseudospectral) methods provide an attractive approach to the numerical simulation of nonlinear partial differential equations (PDEs), combining the flexibility of grid-point evaluation of nonlinearities with exponential convergence for analytic solutions.

Unfortunately, this attraction comes with an important caveat: exponential convergence is not necessarily *rapid* convergence. This problem can be conceptualized from either the physical or spectral point of view. From the physical standpoint, a pseudospectral method has a "built-in" grid that must adequately resolve a function for accurate interpolation. With approaches based on the Fourier Transform (including Chebyshev polynomials, used in this paper), the Nyquist limit of two points per wavelength is a hard limit for band-limited functions, and that extends as a rule-of-thumb [1] to a "few" points per characteristic wavelength for analytic but not band-limited functions.

From the spectral standpoint, these methods converge exponentially for analytic functions that do not have singularities within a characteristic domain on the complex plane. For Fourier-based methods, this is a strip of finite width centred on the real axis, and for Chebyshev polynomial interpolation on the interval $[-1, 1]$, this region is an ellipse with foci at $\pm 1 + 0\mathbf{i}$, where $\mathbf{i}$ is the imaginary unit. The amplitude of the kth mode in the expansion of an analytic function will, for large k, decay exponentially at a rate proportional to the size of the singularity-free area. When the expansion is restricted to finite k in implementation, the unresolved modes are either truncated (for purely spectral methods) or aliased back onto

* Tel.: +1 (514) 421 4282.
E-mail address: csubich@uwaterloo.ca.
[1] Present address: Meteorological Research Division, Environment Canada, 2121 Trans-Canada Highway, Dorval, Québec, H9P 1J3, Canada.

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lower modes (for spectral collocation methods), but the error contributed is exponentially small owing to the exponential convergence rate and low amplitudes near the truncation limit.

Unfortunately, exponential convergence in the large-N asymptote is often still too slow for practical computation. For solutions with shocks and localized, rapid oscillations, resolving the *local* variation sufficiently to take advantage of exponential convergence still requires high *global* grid resolution. This situation is particularly problematic in two and three dimensions, where numerical solutions are often memory-constrained – the very environment where spectral methods are at their best for well-resolved problems.

1.1. Grid adaptation

This limitation can be circumvented by two broad approaches. The first approach modifies the "global" portion of the global spectral method and instead discretizes the computational domain through high-order but ultimately local spectral elements [2,3]. These methods can be resolution-adaptive either through element-splitting (h refinement) or through locally increasing element order (p refinement). They are also highly parallelizable in that information is transferred between elements only at their boundary surfaces – in this regard spectral element methods behave analogously to the classic finite-element approaches. However, in sharing the behaviour of finite element methods, the spectral element approach also loses the extremely convenient ability to evaluate derivatives using $O(N\log(N))$ fast transforms, mostly based on the Fast Fourier Transform.

The second approach, used in this work, is to apply a global spectral method to a transformed computational domain. This approach retains the utility of derivative evaluations based on the Fast Fourier Transform (FFT), and it has been used with success in simulating the Navier–Stokes equations in domains with smooth topography along the boundaries [4]. Adaptive grid mappings with an explicit functional form were constructed in Tee and Trefethen [5] based on modelling the location of function singularities with Padé approximants. However, robustly approximating function singularities is a relatively slow, iterative procedure [6], and applying the adaptivity in service of the time-varying solution to a PDE requires interpolating the solution from the old to the new grid at each adaptation, which at $O(N^2)$ dominates the fast derivative evaluations.

In contrast, this work adapts the idea of a moving mesh partial differential equation (MMPDE), developed for finite difference problems in one dimension by Huang et al. [7]. These methods – since extended to multiple dimensions [8] – extend the underlying PDE by introducing grid movement such that the coordinate is itself a function of an unmoving, computational coordinate (ξ here) and time, giving $x = x(\xi,t)$. The method calculates grid movement ($\dot{x} = x_t(\xi,t)$) by solving an ancillary PDE designed to evenly distribute a measure of discretization error.

This approach was directly applied to pseudospectral methods in Mulholland et al. [9] for elliptic PDEs and Mulholland et al. [10] for time-varying PDEs, but the method still relied on the calculation of a coarse, finite-difference approximation of the discretization error. The grid nodes calculated with the finite-difference approach were then interpolated (with filtering to ensure smoothness) to give the $x(\xi)$ mapping for the pseudospectral discretization. Unfortunately, this approach is sensitive to the resulting filtering parameters, and error monitor functions appropriate for the coarse, finite-difference grid result in unnecessary over-fitting in regions that are already well-resolved.

Instead, it is more useful to consider the grid mapping as a continuous, smooth transformation between the computational coordinate ξ and the physical coordinate x, expressing x as the sum of spectral components. With the Chebyshev polynomials used in this work, this gives $x(\xi,t) = \sum_{k=0}^{N-1} a_k(t)T_k(\xi)$, where $a_k(t)$ is a time-varying coefficient and T_k is the kth Chebyshev polynomial, discussed in Section 2.1. This expansion permits the treatment of the moving mesh partial differential equation in a manner consistent with the physical partial differential equation of interest.

For Fourier-based discretization, the MMPDE method was directly applied to the phase-field equations in two and three dimensions by Feng et al. [11], and this method was extended to two-phase flows in Shen and Yang [12]. In both cases, the arc-length of the phase function in the computational coordinate served to accurately parameterize the discretization error, because the phase function only underwent change at the interface between phases.

This work derives an alternate, more general approach to the estimation of discretization error. Instead of relying on a fixed functional expression such as the arc-length that is deemed to estimate error, the subsequent develops an approach based on spectral filtering, where high-frequency components are considered poorly-resolved and possibly in need of locally higher resolution. This error approximation is filtered for smoothness, giving an approximation appropriate for the MMPDE method.

1.2. Organization

Section 2 introduces the principles of the moving mesh methods used in this work, beginning with a general overview of the approach and extending to the innovations made in this work. The Chebyshev polynomial-based discretization is briefly reviewed in Section 2.1, and Section 2.2 introduces the filtering-based approach used for the error monitor used in this work's approach.

Section 3 presents the results of numerical testing applied to this approach in two parts, where Section 3.1 describes the improved convergence rates for analytically-known functions, both near-singular and smooth, and Section 3.2 describes the

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