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Finite element simulation of eddy current problems using magnetic scalar potentials

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ABSTRACT

We propose a new implementation of the finite element approximation of eddy current problems using, as the principal unknown, the magnetic field. In the non-conducting region a scalar magnetic potential is introduced. The method can deal automatically with any topological configuration of the conducting region and, being based on the search of a scalar magnetic potential in the non-conducting region, has the advantage of making use of a reduced number of unknowns. Several numerical tests are presented for illustrating the performance of the proposed method; in particular, the numerical simulation of a new type of transformer of complicated topological shape is shown.

1. Introduction

Eddy current equations are a well-known approximation of Maxwell equations obtained by disregarding the displacement current term; as a consequence, wave propagation phenomena are not taken into account, and only diffusion of electromagnetic fields is considered.

This is typically the case of “slow” fields, or of low frequency time-harmonic problems, usually appearing in electrotechnics. For instance, induction heating, transformers, magnetic levitation, non-destructive testing, and biomedical identification problems can be modeled by the eddy current equations.

Many papers have been devoted to the numerical simulation of these problems: let us only refer to the book by Alonso Rodríguez and Valli [1] and to the references therein.

As it is well-known, the time-dependent Maxwell equations read:

$$\begin{cases} \dfrac{\partial \mathcal{D}}{\partial t} - \operatorname{curl} \mathcal{H} = -\mathcal{J} & \text{Maxwell–Ampère equation} \\ \dfrac{\partial \mathcal{B}}{\partial t} + \operatorname{curl} \mathcal{E} = 0 & \text{Faraday equation,} \end{cases} \tag{1}$$

where the physical quantities that appear are the magnetic field $\mathcal{H}$, the electric field $\mathcal{E}$, the magnetic induction $\mathcal{B}$, the electric induction $\mathcal{D}$ and the electric current density $\mathcal{J}$.

When the problem is driven by an applied current density $\mathcal{J}_e$, one needs to consider the generalized Ohm law $\mathcal{J} = \boldsymbol{\sigma}\mathcal{E} + \mathcal{J}_e$, where $\boldsymbol{\sigma}$ is the electric conductivity (vanishing in non-conducting regions). Moreover, a linear dependence

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of the form $\mathcal{D} = \boldsymbol{\varepsilon}\mathcal{E}$, $\mathcal{B} = \boldsymbol{\mu}\mathcal{H}$ is usually assumed; here $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are the electric permittivity and magnetic permeability, respectively. In many physical and engineering problems, the region of interest is a non-homogeneous and non-isotropic medium: therefore, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are not scalar constants, but are symmetric and uniformly positive definite matrices (with entries that are bounded functions of the space variable).

The system of equations obtained when the displacement current term $\frac{\partial \mathcal{D}}{\partial t}$ is disregarded is the eddy current system:

$$\begin{cases} \operatorname{curl} \mathcal{H} = \boldsymbol{\sigma}\mathcal{E} + \mathcal{J}_e \\ \boldsymbol{\mu}\dfrac{\partial \mathcal{H}}{\partial t} + \operatorname{curl} \mathcal{E} = 0. \end{cases} \tag{2}$$

The reader interested in a precise mathematical justification of this model is referred to the papers by Alonso [2], Ammari et al. [3], and Costabel et al. [4].

Although the same approach we are going to propose can be used for the time-dependent case, for the sake of simplicity in this paper we prefer to focus on the time-harmonic case, namely, the applied current density $\mathcal{J}_e$ is an alternating current, having the form $\mathcal{J}_e(\mathbf{x}, t) = \mathbf{J}_*(\mathbf{x}) \cos(\omega t + \phi)$, where $\mathcal{J}_*(\mathbf{x})$ is a real-valued vector function, $\omega \neq 0$ is the angular frequency and ϕ is the phase angle. This is equivalent to the representation

$$\mathcal{J}_e(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{J}_*(\mathbf{x}) e^{\imath(\omega t + \phi)}\right] = \mathrm{Re}\left[\mathbf{J}_e(\mathbf{x}) e^{\imath\omega t}\right],$$

where we have introduced the complex-valued vector function $\mathbf{J}_e(\mathbf{x}) := \mathbf{J}_*(\mathbf{x}) e^{\imath\phi}$.

Accordingly, we look for a time-harmonic solution given by

$$\mathcal{E}(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{E}(\mathbf{x}) e^{\imath\omega t}\right], \quad \mathcal{H}(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{H}(\mathbf{x}) e^{\imath\omega t}\right],$$

and the time-harmonic eddy current equations, derived from (2) under these assumptions, read

$$\begin{cases} \operatorname{curl} \mathbf{H} = \boldsymbol{\sigma}\mathbf{E} + \mathbf{J}_e \\ \operatorname{curl} \mathbf{E} + \imath\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0}. \end{cases} \tag{3}$$

Note that for the uniqueness of the electric field in the non-conducting region one needs additional conditions. However we are not interested in the computation of that quantity, hence the reader interested to the complete system can refer to Alonso Rodríguez and Valli [1].

For solving these equations, the most popular approaches are based on vector potentials. Denoting by Ω_C the conducting region and by Ω_I the non-conducting region, the most classical method is that using a vector potential $\mathbf{A}$ of the magnetic induction $\boldsymbol{\mu}\mathbf{H}$ in the whole computational domain Ω and a scalar electric potential V_C in the conducting region Ω_C, satisfying $\operatorname{curl} \mathbf{A} = \boldsymbol{\mu}\mathbf{H}$ in Ω and $-\operatorname{grad} V_C = \mathbf{E}_{|\Omega_C} + \imath\omega\mathbf{A}_{|\Omega_C}$ in Ω_C.

For numerical approximation this approach is rather expensive, as one has to discretize a vector field in the whole domain and a scalar function in the conducting region. Moreover, if classical Lagrange nodal elements are used for the approximation of each single component of $\mathbf{A}$, gauging is compulsory (namely, additional restriction on $\mathbf{A}$ have to be introduced); moreover, the efficiency of the scheme is not guaranteed in the presence of re-entrant corners (see, e.g., Costabel et al. [5]). On the other hand, if edge elements are employed for the approximation of the complete vector field $\mathbf{A}$, a Lagrange multiplier has to be introduced or the resulting linear system turns out to be singular, and in any case it needs special care for being solved. Let us however mention that in this framework one of the most efficient method is that for which gauging is performed by adding a perturbation term $\varepsilon\mathbf{A}$ in the non-conducting region (gauging by "regularization": see Schöberl and Zaglmayr [6], Ledger and Zaglmayr [7]); an important feature of this strategy is that it is well-suited for high-order approximation.

An alternative approach, with a smaller number of unknowns, is to use the formulation in terms of the magnetic field and to introduce a scalar magnetic potential in the non-conducting region, (see, e.g., Bossavit [8], Alonso Rodríguez et al. [9], Bermúdez et al. [10]). An even cheaper way is based on the coupling of the magnetic field in the conducting domain with the flux of the magnetic induction on the interface, thus eliminating the need of considering unknowns inside the non-conducting region: in other words, the magnetic potential in the non-conducting region is eliminated by using potential theory (see, e.g., Bossavit and Vérité [11], Meddahi and Selgas [12]). However, it is worth noting that both approaches have to face difficulties arising from the topology of the non-conducting domain.

In fact, in general topological situations the scalar magnetic potential is a multivalued function. More precisely it is known that if the insulator is not simply-connected there are closed curves contained in Ω_I that are not the boundary of any surface contained in Ω_I; the space of scalar magnetic potentials includes multivalued functions with a constant jump across suitable surfaces "cutting" the non-bounding cycles of Ω_I.

The identification of such surfaces can be a difficult task (pioneering works dealing with this aspect are due to Kotiuga [13–15]; see also the in-depth analysis of the importance of topological issues in electromagnetism presented in Hiptmair [16], Gross and Kotiuga [17]). However it is useful to remark that the key point is that the gradient of a multivalued function jumping on one of these surfaces is a loop field, namely, a curl-free vector field whose line integral on at least one closed curve contained in Ω_I is different from 0 (this closed curve is precisely the non-bounding cycle cut by the surface). For a suitable formulation in terms of a scalar magnetic potential what is really needed are these loop fields.