Journal of Computational Physics 293 (2015) 14–28

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Tempered fractional calculus ☆

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ARTICLE INFO

Article history:
Received 17 February 2014
Received in revised form 3 April 2014
Accepted 7 April 2014
Available online 16 April 2014

Keywords:
Fractional calculus
Anomalous diffusion
Random walk

ABSTRACT

Fractional derivatives and integrals are convolutions with a power law. Multiplying by an exponential factor leads to tempered fractional derivatives and integrals. Tempered fractional diffusion equations, where the usual second derivative in space is replaced by a tempered fractional derivative, govern the limits of random walk models with an exponentially tempered power law jump distribution. The limiting tempered stable probability densities exhibit semi-heavy tails, which are commonly observed in finance. Tempered power law waiting times lead to tempered fractional time derivatives, which have proven useful in geophysics. The tempered fractional derivative or integral of a Brownian motion, called a tempered fractional Brownian motion, can exhibit semi-long range dependence. The increments of this process, called tempered fractional Gaussian noise, provide a useful new stochastic model for wind speed data. A tempered fractional difference forms the basis for numerical methods to solve tempered fractional diffusion equations, and it also provides a useful new correlation model in time series.

1. Introduction

Fractional derivatives were invented by Leibniz soon after the more familiar integer order derivatives [38,52], but have only recently become popular in applications. They are now used to model a wide variety of problems in physics [24,35,38,50,51,55,65], finance [23,27,37,42,60,61], biology [2,4,22,26,36], and hydrology [1,7,8,15,17,62]. Fractional derivatives can be most easily understood in terms of their connection to probability [45,46]. Einstein [20] explained the connection between random walks, Brownian motion, and the diffusion equation $\partial_t p(x,t) = \partial_x^2 p(x,t)$. Sokolov and Klafter [63] review the modern theory of anomalous diffusion, where the integer order derivatives in the diffusion equation are replaced by their fractional analogues: $\partial_t^\beta p(x,t) = \partial_x^\alpha p(x,t)$. A fractional space derivative of order $\alpha < 2$ corresponds to heavy tailed power law particle jumps $\mathbb{P}[J > x] \approx x^{-\alpha}$ (the famous Lévy flight), while a fractional time derivative of order $\beta < 1$ models heavy tailed power law waiting times $\mathbb{P}[W > t] \approx t^{-\beta}$ between jumps. Hence fractional space derivatives model anomalous super-diffusion, where a plume of particles spreads faster than the traditional diffusion equation predicts, and fractional time derivatives model anomalous sub-diffusion.

The goal of this paper is to describe a new variation on the fractional calculus, where power laws are tempered by an exponential factor. This exponential tempering has both mathematical and practical advantages. Mantegna and Stanley

☆ This work was partially supported by NSF grant DMS-1025486, a Fujian Governmental Scholarship, and the National Natural Science Foundation of China (Grant No. 11101344).
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http://dx.doi.org/10.1016/j.jcp.2014.04.024
0021-9991/

[40] proposed a truncated Lévy flight to capture the natural cutoff in real physical systems. Koponen [29] introduced the tempered Lévy flight as a smoother alternative, without a sharp cutoff. Cartea and del-Castillo-Negrete [12] developed the tempered fractional diffusion equation that governs the probability densities of the tempered Lévy flight. Unlike the truncated model, tempered Lévy flights offer a complete set of statistical physics and numerical analysis tools. Random walks with exponentially tempered power law jumps converge to a tempered stable motion [13]. Probability densities of the tempered stable motion solve a tempered fractional diffusion equation that describes the particle plume shape [3], just like the original Einstein model for traditional diffusion. Tempered fractional derivatives are approximated by tempered fractional difference quotients, and this facilitates finite difference schemes for solving tempered fractional diffusion equations [3]. The tempered diffusion model has already proven useful in applications to geophysics [44,70,71] and finance [10,11]. In finance, the tempered stable process models price fluctuations with *semi-heavy tails*, resembling a pure power law at moderate time scales, but converging to a Gaussian at long time scales [5]. Since the anomalous diffusion eventually relaxes into a traditional diffusion profile at late time, this model is also called *transient anomalous diffusion* [71].

Kolmogorov [28] invented a new stochastic model for turbulence in the inertial range. Mandelbrot and Van Ness [39] pointed out that this stochastic process is the fractional derivative of a Brownian motion, and coined the name *fractional Brownian motion*. Fractional Brownian motion can exhibit a very useful property called *long range dependence*, where correlations fall off like a power law with time lag. A new variation on this model, called *tempered fractional Brownian motion*, is the tempered fractional derivative of a Brownian motion [47]. Tempered fractional Brownian motion can exhibit *semi-long range dependence*, with correlations that fall off like a power law at moderate time scales, but then eventually become short-range dependent at long time scales. This extends the Kolmogorov model for turbulence to also include low frequencies, and in fact tempered fractional Brownian motion provides a time-domain stochastic process model for the famous Davenport spectrum of wind speed [6,16,25,53], which is used to design electric power generation facilities.

2. Tempered fractional diffusion

We begin by recalling the connection between random walks, Brownian motion, and the diffusion equation (see [46] for complete details). Given a random walk of mean zero particle jumps $S(n) = X_1 + \cdots + X_n$, the Central Limit Theorem implies that $n^{-1/2}S(nt) \Rightarrow B(t)$ in distribution. The probability density $p(x,t)$ of the Brownian motion limit $B(t)$ solves the diffusion equation $\partial_t p(x,t) = D\partial_x^2 p(x,t)$. This useful connection between Brownian motion, random walks, and the diffusion equation assumes finite variance particle jumps. Power law jumps with density $f(x) = C\alpha x^{-\alpha-1}\mathbf{1}_{[C^{1/\alpha},\infty)}(t)$ for $1 < \alpha < 2$ have a finite mean but an infinite variance. Subtract the mean, and apply the extended central limit theorem [46, Theorem 3.37] to get $n^{-1/\alpha}S(nt) \Rightarrow A(t)$. Now the probability density $p(x,t)$ of the α-stable Lévy motion $A(t)$ solves the fractional diffusion equation $\partial_t p(x,t) = D\partial_x^\alpha p(x,t)$.

Tempered fractional diffusion applies an exponential tempering factor to the particle jump density. Consider a random walk $S^\varepsilon(n)$ with particle jump density

$$f_\varepsilon(x) = C_\varepsilon^{-1} x^{-\alpha-1} e^{-\lambda x}\mathbf{1}_{[\varepsilon,\infty)}(x) \quad \text{where } C_\varepsilon = \int_\varepsilon^\infty x^{-\alpha-1}e^{-\lambda x}dx \tag{1}$$

using the incomplete gamma function, and define the Poisson jump rate

$$\lambda_\varepsilon = D\frac{\alpha}{\Gamma(1-\alpha)}C_\varepsilon \tag{2}$$

for any $\varepsilon > 0$. To ease notation, we begin with the case of positive jumps.

Theorem 2.1. *Suppose $0 < \alpha < 1$. Given a random walk $S^\varepsilon(n) = X_1^\varepsilon + \cdots + X_n^\varepsilon$ with independent jumps, each having probability density function (1), and an independent Poisson process N_t^ε with rate (2), as $\varepsilon \to 0$ we have the convergence*

$$S^\varepsilon\left(N_t^\varepsilon\right) \quad \Rightarrow \quad A(t) \tag{3}$$

where the limit is a tempered stable process whose probability density function $p(x,t)$ has Fourier transform

$$\hat{p}(k,t) = e^{-tD[(\lambda+ik)^\alpha - \lambda^\alpha]} \tag{4}$$

for any $t \geq 0$.

Proof. The Poisson random variable satisfies $\mathbb{P}(N_t^\varepsilon = n) = e^{-\lambda_\varepsilon t}(\lambda_\varepsilon t)^n/n!$ for $n \geq 0$, and then

$$P_\varepsilon(x,t) = \mathbb{P}\left(S^\varepsilon\left(N_t^\varepsilon\right) \leq x\right) = \sum_{n=0}^\infty \mathbb{P}\left(S^\varepsilon(n) \leq x \middle| N_t^\varepsilon = n\right)\mathbb{P}\left(N_t^\varepsilon = n\right)$$

by the law of total probability. Apply the Fourier–Stieltjes transform $\hat{f}(k) = \int e^{-ikx}F(dx)$ where $f(x) = F'(x)$, noting that $\hat{f}_\varepsilon(k)^n$ is the Fourier transform of the probability distribution of $S^\varepsilon(n)$, to get