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Wave–diffusion dualism of the neutral-fractional processes

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ABSTRACT

In this paper, a neutral-fractional equation is introduced and analyzed. In contrast to the general time- and space-fractional diffusion equation, the neutral-fractional equation contains fractional derivatives of the same order α, $1 \le \alpha \le 2$ both in space and in time. As it has been shown earlier, solutions of the neutral-fractional equation can be interpreted as damped waves with the constant propagation velocities that means that this equation inherits some characteristics of the wave equation. Otherwise, the first fundamental solution of the one-dimensional neutral-fractional equation is known to be a spatial probability density function evolving in time and is thus related to the diffusion processes. In this paper, we investigate the entropy and the entropy production rate of the neutral-fractional equation and show that both of them are strongly connected to those of the diffusion processes. Thus a wave–diffusion dualism of the processes described by the neutral-fractional equation is established.

1. Introduction

Within the last few decades a lot of results related both to the theory of the fractional differential equations and their applications in physics, chemistry, engineering, medicine, biology, etc. have been obtained (see e.g. [3,9–17,21,24,25,28,36] to mention only few of many recent publications).

In particular, several models of the anomalous transport processes in the form of the time- and/or space-fractional diffusion–wave equations have been considered by a number of researchers. Anomalous transport processes include the anomalous diffusion (sub- and supper-diffusion), the anomalous wave propagation, and the diffusion–wave processes that were described until now as some intermediate processes between the diffusion and the wave propagation. In this paper, another interpretation of the processes described by some partial differential equations of fractional order is suggested. Namely, we show that they are not a mixture of a diffusion process and a wave propagation, but rather a new phenomena that behaves as a diffusion with respect to some physical characteristics whereas it looks like a wave regarding other characteristics. Thus one can speak about a wave–diffusion dualism of the processes described by these equations that we call the neutral-fractional equations and that were previously referred to as the neutral-fractional diffusion equations or the fractional wave equations depending on what characteristics one was interested in.

The neutral-fractional equations we consider in this paper contain fractional derivatives of the same order α, $1 \le \alpha \le 2$ both in space and in time. The fractional derivative in time is interpreted in the Caputo sense whereas the space-fractional derivative is taken in form of an inverse operator to the fractional Riesz potential (Riesz fractional derivative).

As has been shown in [14], both a maximum location and the "gravity"- and "mass"-centers of the first fundamental solution to the neutral-fractional equation propagate with the constant velocities as the solutions to the wave equation

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($\alpha = 2$), but in contrast to the wave equation these velocities are different from each other for a fixed value of α, $1 < \alpha < 2$. Let us mention that the propagation velocity v of a maximum location of the first fundamental solution to the time-fractional diffusion–wave equation of the order α is determined by the formula (see e.g. [20,22])

$$v(t,\alpha) = C_\alpha t^{\frac{\alpha}{2}-1}.$$

For $1 < \alpha < 2$, the propagation velocity v depends on time t and is a decreasing function that varies from $+\infty$ at time $t = 0+$ to zero as $t \to +\infty$ that makes it difficult to interpret solutions to the time-fractional diffusion–wave equation as some waves.

It is well known that the anomalous transport processes can be modeled in terms of the continuous time random walk processes and described by the time- and/or space-fractional differential equations that are derived from the stochastic models for a special choice of the jump probability density functions with the infinite first or/and second moments (see e.g. [11,17,28]). The neutral-fractional equation can be obtained from the continuous time random walk model, too. In [16], the case of the waiting time probability density function and the jump length probability density function with the same power law asymptotic behavior has been considered. Under some standard assumptions, the neutral-fractional equation can be asymptotically derived from the continuous time random walk model mentioned above (see [16] for details). Thus it is not a surprise that the fundamental solution to the neutral-fractional equation can be interpreted as a probability density function. In this paper, we investigate the entropy and the entropy production rate of the neutral-fractional equation and show that both of them are strongly connected to those of the diffusion processes.

The concept of entropy was first introduced in the macroscopic thermodynamics and then extended for description of some phenomena in statistical mechanics, information theory, ergodic theory of dynamical systems, etc. Historically, many definitions of entropy were proposed and applied in different knowledge areas. In this paper, we employ the statistical concept of entropy that goes back to Shannon and was introduced by him in the theory of communication and transmission of information (see [34]). The entropy of the processes governed by the time- and space-fractional diffusion equations was considered in [10,12] and [30,31], respectively.

From the mathematical viewpoint, the neutral-fractional equation we deal with in this paper was considered for the first time in [5], where an explicit formula for the fundamental solution of the one-dimensional neutral-fractional equation was derived. In [26], a space–time fractional diffusion–wave equation with the Riesz–Feller derivative of order $\alpha \in (0, 2]$ and skewness θ and with the Caputo fractional derivative of order $\beta \in (0, 2]$ was investigated in detail. A particular case of this equation that for $\theta = 0$ corresponds to our one-dimensional neutral-fractional wave equation was mentioned in [26], too. In [29], a fundamental solution to the one-dimensional neutral-fractional equation was deduced and analyzed in terms of the Fox H-function.

In [14], the one-dimensional neutral-fractional equation was investigated from the viewpoint of an interpretation of its solutions as the damped waves. Its fundamental solution was derived in terms of elementary functions for all values of α, $1 \le \alpha < 2$. For the fundamental solution, both its maximum location and its maximum value were determined in closed form as well as the propagation velocities of the maximum location and the "gravity"- and "mass"-centers of the fundamental solution. In [13], a multi-dimensional neutral-fractional equation with a special focus given to the three-dimensional case was analyzed. The fundamental solution to this equation is a spherically symmetric function that possesses some nice integral representations and can be even written down in explicit form in terms of elementary functions in the one- and three-dimensional cases. In contrast to the one-dimensional case, the fundamental solution cannot be interpreted as a probability density function in the two- and three-dimensional cases and thus these equations cannot be employed for modeling of any diffusion processes. Instead, their fundamental solutions can be interpreted as some damped waves with the constant phase velocities that depend only on the order α of the neutral-fractional equation.

In this paper, we mainly deal with the one-dimensional neutral-fractional equation with a special focus given to an interpretation of its solutions as some diffusion processes. For the sake of completeness, the multi-dimensional case and some results regarding an interpretation of the solutions as the damped waves are also mentioned.

The rest of the paper is organized as follows. In Section 2, the basic definitions, problem formulation, and some analytical results for the initial-value problems for the multi-dimensional neutral-fractional equation are presented. In particular, the explicit formulas for the fundamental solutions for the one- and three-dimensional equations are derived in terms of the elementary functions for all values of α, $1 \le \alpha < 2$. In the two-dimensional case, such simple explicit formulas seem to be not available. Section 3 is devoted to a probabilistic interpretation of the solutions to the one-dimensional neutral-fractional equation. In particular, the Shannon entropy and the entropy production rate are calculated. As in the case of the diffusion equation, the entropy production rate decreases with the time as t^{-1}. Surprisingly, the law for the entropy production rate does not depend on the equation order α and it is twice as much as the entropy production rate of the diffusion equation. In Section 4, an interpretation of the solutions to the neutral-fractional equation as damped waves is presented. In particular, we show that the phase velocity of the fundamental solution of the neutral-fractional equation is a constant that depends only on the equation order α. To illustrate analytical findings, some results of numerical calculations, plots, their physical interpretation and discussion are presented. In Section 5, some conclusions and open problems for further research are formulated.