Journal of Computational Physics 293 (2015) 135–141

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Increasing the efficiency of shooting methods for terminal value problems of fractional order

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ARTICLE INFO

Article history:
Received 30 April 2014
Received in revised form 5 September 2014
Accepted 27 October 2014
Available online 29 October 2014

Keywords:
Terminal value problem
Numerical solution
Shooting method
Caputo derivative

ABSTRACT

Shooting methods are a well established tool for the numerical solution of terminal value problems of fractional order. However, they can be computationally quite expensive because of their iterative nature in which (a) each single iteration may be costly, and (b) the number of iterations can be large. In this paper we propose algorithmic strategies for improving the efficiency of such methods. Our strategies are aimed at simultaneously reducing the cost of each iteration and reducing the number of required iterations.

1. Introduction

Terminal value problems involving ordinary differential equations of fractional order nowadays play an important role in the modeling of many phenomena in physics, engineering, etc., both in their own right and as sub-problems arising in connection with fractional-order processes that have been observed at a point in time other than their starting point. The class of problems that we shall consider here is of the form

$$D^{\alpha}_{*a}y(t) = f\big(t, y(t)\big), \qquad y(b) = y^*, \quad a \le t \le b, \tag{1}$$

which is the most general form of a terminal value problem involving an explicit single-term differential equation. Following the commonly accepted notation (see, e.g., [5] and the references cited therein), we use the symbol D^{α}_{*a} in Eq. (1) and throughout the remainder of this paper to denote the Caputo differential operator of order α with starting point a; in particular the subscripted asterisk is used to distinguish this operator from the Riemann–Liouville derivative that is usually designated by D^{α}_{a} and that we will not consider here. We restrict our attention to the case that $0 < \alpha < 1$ because this is the case that is relevant to the vast majority of applications, cf., e.g., [1,2,5,6,12,19] and the references cited therein. Thus, we are looking for a solution $y : [a, b] \to \mathbb{R}$ to a fractional differential equation involving a Caputo differential operator D^{α}_{*a} with starting point a (the left end point of the interval where the solution is sought) under the assumption that the value of the function y is known only at the point b, i.e. the right end point of the interval of interest. The question for the existence and uniqueness of solutions to such problems under the usual assumptions (essentially, continuity of f on a suitable set and a Lipschitz property of f with respect to the second variable) has recently been answered affirmatively [4,8,14,15].

Problems of the form (1) arise in a number of situations, in particular when fractional differential equations are used for modeling concrete phenomena in physics, engineering, biology, etc. (cf. [2,5,6,12,19] for specific examples) and information

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http://dx.doi.org/10.1016/j.jcp.2014.10.054
0021-9991/

about the state of the system is known at some point $t = b$, but not at the beginning of the process, i.e. at $t = a$. One is then often interested in finding out how the system started which means that Eq. (1) needs to be solved. Depending on the specific problem at hand, it may be sufficient to find this solution and thus the behavior on the interval $[a, b]$, or one must use this information — in particular the starting value $y(a)$ — as an intermediate result to define a classical initial value problem consisting of this initial condition and the differential equation from (1) and then solve this initial value problem on the larger interval $[a, \tilde{b}]$ with $\tilde{b} > b$, thus not only reconstructing the behavior of the process between the starting point a and the point of observation b, but also predicting its development for future times $t > b$.

We shall therefore now briefly review existing approaches for numerically solving the problem (1) and then propose a technique for increasing their efficiency. In passing, we note that our methods may be used as a building block for solving the even more general problem of finding a solution to Eq. (1) under the additional assumption that the starting point a is also unknown. To the best of our knowledge, this latter problem has not been addressed in the literature yet; we defer a detailed discussion to a later paper.

2. Numerical approaches

Essentially, there are two different ways in which one can tackle the given terminal value problem (1) numerically. The first idea is based on the observation [5, Theorem 6.18] that the problem is essentially a special case of a boundary value problem and can hence be rewritten in the form of the equivalent integral equation of Fredholm type

$$y(t) = y^* + \frac{1}{\Gamma(\alpha)} \int_a^b G(t,s) f\big(s, y(s)\big) \mathrm{d}s \tag{2}$$

where

$$G(t,s) = \begin{cases} -(b-s)^{\alpha-1} & \text{for } s > t, \\ (t-s)^{\alpha-1} - (b-s)^{\alpha-1} & \text{for } s \le t. \end{cases} \tag{3}$$

This approach allows to employ classical techniques like finite elements or finite differences for the solution of such Fredholm equations in order to solve the terminal value problem. However, such an approach has in the past been used mainly for two-point boundary value problems containing differential equations of order $\alpha \in (1, 2)$; cf., e.g., [13,18,20].

In this paper, we shall concentrate on a different method that has been studied in detail, e.g., in [14,16,17]. This approach is also based on a suitable adaptation of a technique known for integer-order problems, namely the idea of a shooting method. When transferred to the fractional-order setting under consideration here, this method can be summarized as follows:

1. We are looking for a solution on the interval $[a, b]$. If the value of the solution y at the starting point a of the differential operator D^{α}_{*a} were known, we would be dealing with a classical initial value problem and could use the corresponding numerical methods.
2. As the value $y(a)$ is not known, we begin by guessing a first approximation y_{10}, say, for $y(a)$.
3. Then we go on in an iterative manner that can be described in an abstract way (that will be made concrete in the ensuing paragraphs) as follows:
   (a) In the k-th passage through the iteration loop, we determine a numerical solution y_k to the initial value problem consisting of the given differential equation from (1) combined with the initial condition $y_k(a) = y_{k0}$. The terminal condition that appears in Eq. (1) is ignored at this stage.
   (b) We then look at the value $y_k(b)$ and compare it to the value y^* required by the terminal condition. If these two values are sufficiently close to each other, we exit the loop and accept y_k as the approximate solution to the complete terminal value problem (1).
   (c) Otherwise, we compute a new starting value $y_{k+1,0}$ from the previous value y_{k0}, taking into account the difference between $y_k(b)$ and y^*, and continue with the iteration.

In their detailed comparative analysis [17], Ford et al. concluded that the Adams-type predictor–corrector method from [9,10] was a very good choice for solving the initial value problems arising in each iteration of the process outlined above. For the purposes of this paper, we shall follow their advice and combine this with the idea from [3] where it has been stated that a suitable choice of the number of corrector iterations in the predictor–corrector method improves the convergence order to $O(h^2)$ essentially without increasing the computational cost. However, it will be immediately clear that our key findings indicated below should remain valid for differently chosen numerical solvers.

In order to begin the process, we need to find an initial guess y_{10} for the starting value. Owing to the lack of other information, we here suggest to use the terminal value, i.e. to define

$$y_{10} := y^* \tag{4}$$

where y^* is the given value from the original terminal value problem (1).