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A multi-domain spectral method for time-fractional differential equations

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ABSTRACT

This paper proposes an approach for high-order time integration within a multi-domain setting for time-fractional differential equations. Since the kernel is singular or nearly singular, two main difficulties arise after the domain decomposition: how to properly account for the history/memory part and how to perform the integration accurately. To address these issues, we propose a novel hybrid approach for the numerical integration based on the combination of three-term-recurrence relations of Jacobi polynomials and high-order Gauss quadrature. The different approximations used in the hybrid approach are justified theoretically and through numerical examples. Based on this, we propose a new multi-domain spectral method for high-order accurate time integrations and study its stability properties by identifying the method as a generalized linear method. Numerical experiments confirm *hp*-convergence for both time-fractional differential equations and time-fractional partial differential equations.

1. Introduction

Fractional calculus and the modeling of a variety of non-classic phenomena using fractional differential equations is emerging as an area of substantial activity across many applications in the natural and social sciences (cf. [20,19]). The primary advantage of such models is the introduction of a parameter, α, which can be used to model non-Markovian behavior of spatial or temporal processes. During the last decade, this approach has emerged as generalizations of many classic problems in mathematical physics, including the fractional Burgers' equation [14,29], the fractional Navier–Stokes equation [8,7], the fractional Maxwell equation [13], the fractional Schrödinger equation [11], the fractional Ginzburg Landau equation [11,25], etc. In parallel, numerical methods for classical differential equations, such as finite difference methods [18,17,26], finite element methods [4], spectral methods [16,3,15], and discontinuous Galerkin methods [22,27], have been developed, albeit this remains a relatively new topic of research.

In this work we focus on time-fractional differential equations (TFDE) where the time-fractional derivative emerges as an integro-differential operator, defined by the convolution of the classical derivative of the function and a singular kernel of fractional power-law type. Hence, the solution to a TFDE at a certain time depends on the total history of the solution at previous times and seeks to model problems with memory terms, among other things. Considering the design of numerical schemes, an immediate consequence of this is that the whole trajectory of the numerical solution must be carried forward

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and used in the computation at the current time step. This impacts both the storage and the cost of the numerical method, both of which may substantially increase over time.

Most past work on the development of methods for TFDEs has focused on lower order methods in which a finite difference approach is used to replace the integrant with its forward, backward, or mid-point rules. More recently, a piecewise-linear, discontinuous Galerkin method was proposed in [22] for the fractional wave equation.

Though only sparsely explored in this context, high order methods have the potential to reduce the storage requirement and computational cost, by allowing the use of many fewer points while achieving the same accuracy as that of lower order methods. However, accurate high-order approximation to fractional operators requires a delicate treatment of special functions and integral transforms, similar to what has been developed in the context of spectral methods based on classic orthogonal polynomials [12,6]. In [9], a pseudo-spectral method for fractional differential equations was proposed. Using spectral methods for the TFDE, [16] proposes a spectral-Galerkin method and shows that by taking advantage of special properties of the time-fractional differential operator, positive definite linear systems result, leading to an efficient solver. In [3,2], the authors explored analytical results of the fractional derivative of shifted Jacobi polynomials and discussed two effective approaches – spectral collocation methods and spectral tau methods – for the fractional differential equations and time fractional diffusion-wave equations. Recently, [15] derived three-term-recurrence relations for fractional integrals and derivatives for the Jacobi polynomials, and provided an elegant collocation approach that can be seen as a generalization of the classical numerical differentiation. A slightly different but related approach is developed in [30,31] where a Petrov–Galerkin spectral element method is proposed for fractional ODEs, defined through the Riemann–Liouville definition. Unlike [31], in this paper we focus on a collocation form that is more suitable for variable coefficients and nonlinear source functions.

The majority of past developments of high-order methods for the TFDE emphasize spectral approximations in a single domain. In this paper, we adopt the idea of a multi-domain spectral approach, leading to a generalized high-order linear method with a long tail of past solutions. A similar approach is recently proposed in [28]. There are two challenges that arise out of this approach. First, one needs to discretize the fractional derivative which has a singular kernel within the current element. For this, we rely on the use of the three-term-recurrence approach proposed in [15]. The second task, which appears more challenging, is to compute the history part accurately and efficiently, i.e., to compute the integral over previous elements with high precision. In order to deal with this, we first extend the three-term-recurrence relation in [15] to work on all elements and then combine it with a Gaussian quadrature for terms far away from the current element. This hybrid approach is needed to control stability issues related to the recurrence relation. Values of these integrals together with the source function form the right hand side of the linear system.

To understand the stability properties of this approach, we carry out a stability study based on a companion matrix approach and verify *hp*-convergence of our method for time-fractional differential equations and time-fractional partial differential equations. The paper seeks to establish a computational framework for high-order accurate integration of the TFDE, and our work clearly indicates that the multi-domain spectral approach, as an optimized combination of low and high order methods, can provide excellent performance for numerical simulations of problems dominated by memory effects.

What remains of the paper is organized as follows. Basic notation of fractional calculus and TFDE are provided in Section 2. In Section 3, we describe the core techniques of our spectral approximation to the fractional derivative. The algorithmic details and the stability analysis of the hybrid multi-domain spectral method are discussed in Section 4. Numerical results for a multi-term fractional differential equation are provided in Section 5 and Section 6 contains a few concluding remarks.

2. Fractional differential equations

Let us denote the time domain $(0,T)$ as Ω, and let $t \in \Omega$. The fractional integral of order α of a given function $u(t)$ is defined as

$$ {}_0D_t^{-\alpha}u(t) \triangleq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\mathrm{d}s, \tag{2.1}$$

where $\Gamma(x)$ is the Gamma function. This allows us to define the Caputo fractional derivative of order α $(0<\alpha\le 1)$ as

$$ {}_0D_t^{\alpha}u(t) \triangleq \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\mathrm{d}u(s)}{\mathrm{d}s}\mathrm{d}s, \tag{2.2}$$

which is preferred over alternative definitions to deal with general initial conditions. This definition can naturally be extended to higher order as

$$ {}_0D_t^{\alpha}u(t) \triangleq {}_0D_t^{\alpha-n}\frac{\mathrm{d}^n u(t)}{\mathrm{d}t^n} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-1-\alpha}\frac{\mathrm{d}^n u(s)}{\mathrm{d}s^n}\mathrm{d}s, \tag{2.3}$$

where n is an integer such that $\alpha \in (n-1, n]$.