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Time-stepping error bounds for fractional diffusion problems with non-smooth initial data

William McLean [a,*,1,2], Kassem Mustapha [b,1]

[a] School of Mathematics and Statistics, The University of New South Wales, Sydney 2052, Australia

[b] Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, 31261, Saudi Arabia

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ABSTRACT

We apply the piecewise constant, discontinuous Galerkin method to discretize a fractional diffusion equation with respect to time. Using Laplace transform techniques, we show that the method is first order accurate at the nth time level t_n, but the error bound includes a factor t_n^{-1} if we assume no smoothness of the initial data. We also show that for smoother initial data the growth in the error bound as t_n decreases is milder, and in some cases absent altogether. Our error bounds generalize known results for the classical heat equation and are illustrated for a model problem.

1. Introduction

Consider an initial-value problem for an abstract, time-fractional diffusion equation [7, p. 84]

$$\partial_t u + \partial_t^{1-\nu} Au = 0 \quad \text{for } t > 0, \text{ with } u(0) = u_0 \text{ and } 0 < \nu < 1. \tag{1}$$

Here, we think of the solution u as a function from $[0, \infty)$ to a Hilbert space $\mathcal{H}$, with $\partial_t u = u'(t)$ the usual derivative with respect to t, and with

$$\partial_t^{1-\nu} u(t) = \frac{\partial}{\partial t} \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} u(s)\, ds$$

the Riemann–Liouville fractional derivative of order $1-\nu$. The linear operator A is assumed to be self-adjoint, positive-semidefinite and densely defined in $\mathcal{H}$, with a complete orthonormal eigensystem $\phi_1, \phi_2, \phi_3, \ldots$. We further assume that the eigenvalues of A tend to infinity. Thus,

$$A\phi_m = \lambda_m \phi_m, \quad \langle \phi_m, \phi_n \rangle = \delta_{mn}, \; 0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots,$$

where $\langle u, v \rangle$ is the inner product in $\mathcal{H}$; the corresponding norm in $\mathcal{H}$ is denoted by $\|u\| = \sqrt{\langle u, u \rangle}$. In particular, we may take $Au = -\nabla^2 u$ and $\mathcal{H} = L_2(\Omega)$ for a bounded spatial domain Ω, with u subject to homogeneous Dirichlet or Neumann boundary conditions on $\partial\Omega$. Our problem (1) then reduces to the classical heat equation when $\nu \to 1$.

* Corresponding author.
E-mail addresses: w.mclean@unsw.edu.au (W. McLean), kassem@kfupm.edu.sa (K. Mustapha).

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Many authors have studied techniques for the time discretization of (1), but obtaining sharp error bounds has proved challenging. In studies of explicit and implicit finite difference schemes [1,3,8,14,17,20] the error analyses typically assume that the solution $u(t)$ is sufficiently smooth, including at $t = 0$, which amounts to imposing compatibility conditions on the initial data and source term. In our earlier work on discontinuous Galerkin (DG) time-stepping [11,15,16], we permitted more realistic behaviour, allowing the derivatives of $u(t)$ to be unbounded as $t \to 0$, but were seeking error bounds that are uniform in t using variable time steps. In the present work, we again consider a piecewise-constant DG scheme but with a completely different method of analysis that leads to sharp error bounds even for non-smooth initial data, at the cost of requiring a constant time step Δt. Our previous analysis [11, Theorem 5] of the scheme (5), in conjunction with relevant estimates [10] of the derivatives of u, shows, in the special case of uniform time steps, only a sub-optimal error bound at the nth time level $t_n = n\Delta t$,

$$\|U^n - u(t_n)\| \le C\Delta t^{r\nu}\|A^r u_0\| \quad \text{for } 0 \le r < 1/\nu. \tag{2}$$

In our main result, we substantially improve on (2) by showing that

$$\|U^n - u(t_n)\| \le C t_n^{r\nu-1}\Delta t\|A^r u_0\| \quad \text{for } 0 \le r \le \min(2, 1/\nu). \tag{3}$$

Thus, for a general $u_0 \in \mathcal{H}$ the error is of order $t_n^{-1}\Delta t$ at $t = t_n$, so the method is first-order accurate but the error bound includes a factor t_n^{-1} that grows if t_n approaches zero, until at $t = t_1$ the bound is of order $t_1^{-1}\Delta t = 1$. However, if $1/2 \le \nu < 1$ and u_0 is smooth enough to belong to $D(A^{1/\nu})$, the domain of $A^{1/\nu}$, then the error is of order Δt, uniformly in t_n. For $0 < \nu < 1/2$, no matter how smooth u_0 a factor $t_n^{2\nu-1}$ is present. To the best of our knowledge, only Cuesta et al. [2] and McLean and Thomée [12, Theorem 3.1] have hitherto investigated the time discretization of (1) for the interesting case when the initial data might not be regular, the former using a finite difference–convolution quadrature scheme and the latter a method based on numerical inversion of the Laplace transform.

In the present work, we do not discuss the spatial discretization of (1). By contrast, Jin, Lazarov and Zhou [6] studied the piecewise linear finite element solution $u_h(t) \approx u(t)$ using a quasi-uniform partition of the spatial domain Ω into elements with maximum diameter h, but with no time discretization. They worked with an alternative formulation of the fractional diffusion problem,

$$\partial_{t,\mathrm{C}}^{\nu} u - \nabla^2 u = 0 \quad \text{for } x \in \Omega \text{ and } 0 < t \le T, \tag{4}$$

where $\partial_{t,\mathrm{C}}^{\nu}$ denotes the Caputo fractional derivative of order ν, and proved [6, Theorems 3.5 and 3.7] that if $u_h(0) = u_{0h}$, for an appropriate $u_{h0} \approx u_0$, then

$$\|u_h(t) - u(t)\| + h\|\nabla(u_h - u)\| \le Ct^{\nu(r-1)} \times \begin{cases} h^2\ell_h\|A^r u_0\|, & r \in \{0, 1/2\}, \\ h^2\|A^r u_0\|, & r = 1, \end{cases}$$

where $\ell_h = \max(1, \log h^{-1})$. These estimates for the spatial error complement our bounds for the error in a time discretization.

We define a piecewise-constant approximation $U(t) \approx u(t)$ by applying the DG method [11,13],

$$U^n - U^{n-1} + \int_{t_{n-1}}^{t_n} \partial_t^{1-\nu} AU(t)\,dt = 0 \quad \text{for } n \ge 1, \text{ with } U^0 = u_0, \tag{5}$$

where $U^n = U(t_n^-) = \lim_{t\to t_n^-} U(t)$ denotes the one-sided limit from below at the nth time level. Thus, $U(t) = U^n$ for $t_{n-1} < t \le t_n$. Since we do not consider any spatial discretization, U is a semidiscrete solution with values in $\mathcal{H}$. A short calculation reveals that

$$\int_{t_{n-1}}^{t_n} \partial_t^{1-\nu} AU(t)\,dt = \Delta t^{\nu} \sum_{j=1}^{n} \beta_{n-j} AU^j,$$

with

$$\beta_0 = \Delta t^{-\nu} \int_{t_{n-1}}^{t_n} \frac{(t_n - t)^{\nu-1}}{\Gamma(\nu)}\,dt = \frac{1}{\Gamma(1+\nu)}$$

and, for $j \ge 1$,

$$\beta_j = \Delta t^{-\nu} \int_{t_{n-j-1}}^{t_{n-j}} \frac{(t_n - t)^{\nu-1} - (t_{n-1} - t)^{\nu-1}}{\Gamma(\nu)}\,dt = \frac{(j+1)^{\nu} - 2j^{\nu} + (j-1)^{\nu}}{\Gamma(1+\nu)}.$$