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Damage and fatigue described by a fractional derivative model

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ABSTRACT

As in [1], damage is associated with fatigue that a material undergoes. In this paper, because we work with viscoelastic solids represented by a fractional model, damage is described by the order of the fractional derivative, which represents the phase field satisfying Ginzburg–Landau equation, which describes the evolution of damage.

Finally, in our model, damage is caused, not only by fatigue, but also directly by a source related to environmental factors and described by a positive time function.

1. Introduction

Fatigue as a phenomenon in the material systems, produces a gradual and progressive process of mechanical deterioration and damage involving crack nucleation, creep rupture and then rapid fracture (see [3,7,13,17,18]). Since during damage processes, we observe a change of the internal structure, we follow the view point presented in [1] (see also [11,12]), so we describe such structural variation by an order parameter denoted by $\alpha \in [0,1]$. Thus, we present a model for the study of fatigue and damage by a constitutive equation containing the parameter α, related with the notion of fractional derivative. It is well known that inside the fractional models, a linear viscoelastic body $\mathcal{B}$, can be defined by

$$\tilde{\sigma}(x,t) = \mathbf{A}(x)\,{}^{C}_{t_0}D^{\alpha}_{t}\varepsilon(x,t), \quad \alpha \in [0,1] \tag{1.1}$$

where $\sigma(x,t)$ and $\varepsilon(x,t)$ are the stress and strain tensors respectively, defined in the material point $x \in \mathcal{B} \subset \mathbb{R}^3$ and the time $t \in [0,T)$. While the fourth order tensor $\mathbf{A}(x)$ is assumed positive defined. Finally, the operator ${}^{C}_{t_0}D^{\alpha}_{t}$ denotes the Caputo derivative of α-order (see [4–6]), defined by

$${}^{C}_{t_0}D^{\alpha}_{t}\varepsilon(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{\dot{\varepsilon}(x,\tau)}{(t-\tau)^{\alpha}} d\tau \tag{1.2}$$

where $\Gamma(\cdot)$ is the gamma function and $t_0 < 0$ is a suitable fixed time.

Because of fatigue effects, it occurs a material damage, which we describe by a variation of the coefficient $\alpha(x,t)$ of the fractional derivative [16]. In this framework, the virgin material, supposed as an elastic body, is described by the coefficient $\alpha = 0$. Then, in this pattern, fatigue action produces an increase of damage and of coefficient $\alpha(x,t)$ accordingly.

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Therefore, the material show viscous effects, which will increase with the increments of α, until reaching a limit value $\alpha_C(x) < 1$, wherein the stress goes to zero, with the subsequent fracture [2]. Then in this pattern, the fraction α is a function of time t able to represent the evolution of the material system. For this reason, in addition to the equation of motion, we have to consider a new equation for the $\alpha(x,t)$ variable [12].

Following the view point studied in [1], for this pattern we use the Ginzburg–Landau (G–L) equation, which is able to describe the material structural variations by a phase field or order parameter [9], which in this framework is given by the field $\alpha(x,t)$. Moreover, we introduce a second fatigue effect due to the life time of the material, and to the environmental factors. Indeed in this case the damage is the consequence of the effect of time only, resulting in aging of the material [14].

Hence, in this paper, first we introduce the differential system defined by the motion and the (G–L) equations. Hence, for this model we obtain the thermodynamic consistence. Then, we prove a maximum theorem, by means of which, we prove the asymptotic behavior of the parameter α towards the limit value α_C. Finally, we show some mathematical simulations, which confirms that fatigue effects involve material fractures, after a set of loading and unloading procedures.

2. Constitutive equations and fatigue

For the description of fatigue phenomena and of the resulting damage, we need to specify the essential variables of the mechanical system. For this purpose, we suppose that the state $s(x,t)$ of the body $\mathcal{B}$ is given by the triple

$$s(x,t) = \big(\varepsilon^t(x), \alpha^t(x), \nabla\alpha(x,t)\big) \tag{2.1}$$

where ε^t and α^t denote the histories of the strain tensor at the time t

$$\varepsilon^t(x,\tau) = \varepsilon(x, t-\tau), \quad \tau \in [0,\infty)$$

and of the order parameter at the time t

$$\alpha^t(x,\tau) = \alpha(x, t-\tau), \quad \tau \in [0,\infty).$$

Then we define the mechanical process $P : [0, d_P) \to Sym(\mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}^3$, of duration $d_P < \infty$, defined for any $x \in \mathcal{B}$ and time $t \in [0, d_P)$ by (see [15])

$$P(x,t) = \big(\dot\varepsilon_P(x,t), \dot\alpha_P(x,t), \nabla\alpha_P(x,t)\big)$$

where the dot denotes the time derivative. Moreover, the response of the material is defined by the function $E(t)$ defined by the pair

$$E(x,t) = \big(\sigma(x,t), \mathcal{F}(x,t)\big)$$

Here, the stress is given by

$$\sigma(t) = \frac{1}{4}\big(\alpha_C - \alpha(t)\big)^2 \mathbf{A}(x) {}_{t_0}^{C}D_t^{\alpha} \varepsilon(x,t) \tag{2.2}$$

where the operator ${}_{t_0}^{C}D_t^{\alpha}\varepsilon(x,t)$ denotes the Caputo fractional derivative (1.2).

The coefficient $\alpha_C < 1$ is such that $0 \le \alpha(t) \le \alpha_C$, when $\alpha(t) = \alpha_C$ we have fracture. Moreover, the function $\mathcal{F}(t)$ that describes fatigue, is defined by

$$\mathcal{F}\big(s(t_0), P_{[0,t)}\big) = \int_{t_0}^{t} \big(\alpha_C - \alpha(\tau)\big)\mathbf{A}(x) {}_{t_0}^{C}D_t^{\alpha}\varepsilon(x,\tau) \cdot \dot\varepsilon(x,\tau)\, d\tau \tag{2.3}$$

Here $s(t_0)$ denotes the initial virgin state and $P_{[0,t)}$ denotes the process restricted to the interval $[0,t)$.

The differential system describing fatigue phenomenon is given by the motion equation

$$\rho_0(x)\dot{\mathbf{v}}(x,t) = \nabla \cdot \sigma(x,t) + \rho_0(x)\mathbf{b}(x,t) \tag{2.4}$$

where $\mathbf{v}$ is the velocity, ρ_0 is the density and $\mathbf{b}$ is the external source. Moreover, in this paper we confine our research to deformations for which we can suppose that $\dot{\mathbf{v}}(x,t) = \frac{\partial \mathbf{v}(x,t)}{\partial t}$. As we observed, together with Eq. (2.4), we consider the equation for the phase field or order parameter, identified with the order of the fractional derivative, defined in (1.2) and denoted with $\alpha(x,t)$. Then, we introduce the functions

$$F(\alpha) = \alpha_C - \breve\alpha, \qquad G(\alpha) = 8\alpha_C\left(\breve\alpha^2 - \frac{3}{4}\breve\alpha^2\right) \tag{2.5}$$

where the variable $\breve\alpha$ is given by

$$\breve\alpha = \begin{cases} 0 & \text{if } \alpha < 0 \\ \alpha & \text{if } 0 \le \alpha \le \alpha_C \\ \alpha_C & \text{if } \alpha > \alpha_C \end{cases}$$