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A spreading drop of shallow water

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ABSTRACT

The theoretical solutions and corresponding numerical simulations of Schär and Smolarkiewicz (1996) [3] are revisited. The original abstract problem of a parabolic, slab-symmetric drop of shallow water spreading under gravity is extended to three spatial dimensions, with the initial drop defined over an elliptical compact support. An axisymmetric drop is considered as a special case. The elliptical drop exhibits enticing dynamics, which may appear surprising at the first glance. In contrast, the evolution of the axisymmetric drop is qualitatively akin to the evolution of the slab-symmetric drop and intuitively obvious. Besides being interesting per se, the derived theoretical results provide a simple means for testing numerical schemes concerned with wetting–drying areas in shallow water flows. Reported calculations use the *libmpdata++*, a recently released free/libre and open-source software library of solvers for generalized transport equations. The numerical results closely match theoretical predictions, demonstrating strengths of the nonoscillatory forward-in-time integrators comprising the *libmpdata++*.

1. Introduction

The "shallow water equations" (SWE) is a popular topic in the Journal of Computational Physics (JCP). According to the Thomson Reuters Web of Science, nearly 300 papers on SWE have been published in JCP since 1971 ($\approx 4\%$ of all recorded SWE papers published since 1969), with the accrued citation count approaching 7000 ($\approx 6\%$ of all recorded citations). The SWE attract interdisciplinary interests, as substantiated by numerous contributions in the areas of applied mathematics, astrophysics, environmental engineering, fluid dynamics, geophysics, meteorology and oceanography. Apart from providing a practical long-wave approximation for PDE systems governing, in general, continuously-stratified 3D flows [1,2], SWE also provide a convenient testing ground for development of numerical methods and complex models in a broad range of computational science and engineering.

The work [3] is a representative example amid those using SWE to prove new developments. In [3] the authors addressed the compatibility of Eulerian and Lagrangian numerical integrations of coupled transport equations, in the context of geophysical flows. One of the computational testbeds employed to quantify the accuracy of the proposed schemes was an idealized problem of a 2D slab-symmetric drop of shallow water spreading under gravity. While directly addressing the wetting–drying areas for the SWE, this problem also epitomizes inflating–collapsing material layers in isentropic/isopycnic models for atmospheric/oceanic circulations. Technically, such models resemble a stack of shallow water models, coupled by

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the hydrostacy in the vertical direction [3,4]. Just like in conservative integrations of SWE, diagnosing velocity in material layers may require calculating ratios of the momentum and the layer depth. Preventing this division from becoming ill-defined requires the momentum and the layer depth to vanish compatibly; i.e., with their ratio always being well defined, as implied by the evolutionary (Lagrangian) form of the specific momentum equation (velocity).

Here, we extend the theoretical solutions of [3] to 3D drops with an elliptical compact support, the axisymmetric case of which is described by particularly concise and elegant formulae. The theoretical results provide simple means of quantifying the accuracy of multidimensional schemes required to deal with wetting–drying areas and inflating–collapsing material layers. Theoretical predictions are supplemented with numerical integrations of partial differential SWE using the MPDATA-based nonoscillatory forward-in-time methods, also used in [3] and widely documented in the literature; see [5] for a recent summary. All simulations were performed using *libmpdata++*, a free/libre and open-source software library of solvers for generalized transport equations [6]. The corresponding theoretical and computational results corroborate each other.

The paper is organized as follows. Section 2 presents the problem, summarizes analytical derivations, and discusses theoretical results. Section 3 highlights numerical integrators and compares theoretical and computational results. Remarks in Section 4 conclude the paper.

2. Analytical results

2.1. Problem statement

Following [3], the addressed physical scenario consists of an initially stagnant lenticular drop of a homogeneous incompressible inviscid fluid that spreads under gravity on the plane, in absence of friction and background rotation.[1] The depth-integrated governing PDE postulate the shallow-water (hydrostatic) approximation, physically sound when the width of the drop significantly exceeds its height. Conforming to the formulation in [3], the extended 2D SWE take the dimensionless form

$$\partial_t h + \partial_x(uh) + \partial_y(vh) = 0, \tag{1}$$

$$\partial_t(uh) + \partial_x(uuh) + \partial_y(vuh) = -h\partial_x h, \tag{2}$$

$$\partial_t(vh) + \partial_x(uvh) + \partial_y(vvh) = -h\partial_y h, \tag{3}$$

where h, (u, v), (x, y) and t have the usual meaning of height, velocity components, Cartesian coordinates and time. The indicated variables are normalized, respectively, by a characteristic height scale h_o, the celerity $u_o = (gh_o)^{1/2}$ (with g denoting the gravitational acceleration), a characteristic width scale a, and the characteristic time scale $t_o = a/u_o$.

In contrast to the 1D formulation of [3] — in turn inspired by the study of a SWE ribbon on a rotating plane [7] — our initial drop is defined over an elliptical compact support on the xy plane

$$\begin{aligned} &h(x, y, t=0) > 0 \quad \text{if} \quad \frac{x^2}{\lambda_{x0}^2} + \frac{y^2}{\lambda_{y0}^2} < 1, \\ &h(x, y, t=0) = 0 \quad \text{otherwise}, \end{aligned} \tag{4}$$

where λ_{x0} and λ_{y0} are the major and minor semi-axes of the ellipse. Following the earlier works [3,7,8], the solutions are sought to maintain parabolic profiles in the x and y cuts, so the evolving height can be written as

$$\begin{aligned} &h(x, y, t) = A(\lambda_x, \lambda_y)\left(1 - \frac{x^2}{\lambda_x^2} - \frac{y^2}{\lambda_y^2}\right) \quad \text{if} \quad \frac{x^2}{\lambda_x^2} + \frac{y^2}{\lambda_y^2} < 1, \\ &h(x, y, t) = 0 \quad \text{otherwise}, \end{aligned} \tag{5}$$

where $\lambda_x(t)$ and $\lambda_y(t)$ depend solely on time, while the actual height of the drop at the origin $A(\lambda_x, \lambda_y) > 0\ \forall\ t < \infty$. The theoretical results for the general case of (5) will be presented next, whereas the special axisymmetric case, $\lambda_x = \lambda_y\ \forall\ t$, will be presented in Section 2.3.

2.2. Elliptical drops

Along the $y = 0$ and $x = 0$ symmetry axes the solution is expected to retain the key functional dependencies of the 1D case [3]. Consequently, we postulate $A(\lambda_x, \lambda_y) = \lambda_x^n \lambda_y^n$ in (5), where the exponent $n < 0$ is yet to be determined. Furthermore, it is also assumed that the velocities change linearly across the drop and satisfy $u(\lambda_x, t) = d\lambda_x/dt \equiv \dot{\lambda}_x$, and $v(\lambda_y, t) = d\lambda_y/dt \equiv \dot{\lambda}_y$ at the x and y edges of the drop, respectively. The simplest form that fulfills these requirements is the irrotational flow

[1] Surface tension effects are irrelevant for applications addressed in this paper.