Journal of Computational Physics 289 (2015) 129–148

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A numerical strategy to discretize and solve the Poisson equation on dynamically adapted multiresolution grids for time-dependent streamer discharge simulations

Max Duarte [a,b,*], Zdeněk Bonaventura [c], Marc Massot [d,e,f], Anne Bourdon [d,e,1]

[a] Univ. Nice Sophia Antipolis, CNRS, LJAD, UMR 7351, 06100 Nice, France
[b] CCSE, Lawrence Berkeley National Laboratory, 1 Cyclotron Rd. MS 50A-1148, 94720 Berkeley, CA, USA
[c] Department of Physical Electronics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
[d] CNRS, UPR 288 Laboratoire EM2C, Grande Voie des Vignes, 92295 Châtenay-Malabry, France
[e] Ecole Centrale Paris, Grande Voie des Vignes, 92295 Châtenay-Malabry Cedex, France
[f] Fédération de Mathématiques de l'Ecole Centrale Paris, FR CNRS 3487, France

ARTICLE INFO

Article history:
Received 22 February 2014
Received in revised form 19 February 2015
Accepted 22 February 2015
Available online 27 February 2015

Keywords:
Poisson equation
Multiresolution finite volume scheme
Streamer discharges

ABSTRACT

We develop a numerical strategy to solve multi-dimensional Poisson equations on dynamically adapted grids for evolutionary problems disclosing propagating fronts. The method is an extension of the multiresolution finite volume scheme used to solve hyperbolic and parabolic time-dependent PDEs. Such an approach guarantees a numerical solution of the Poisson equation within a user-defined accuracy tolerance. Most adaptive meshing approaches in the literature solve elliptic PDEs level-wise and hence at uniform resolution throughout the set of adapted grids. Here we introduce a numerical procedure to represent the elliptic operators on the adapted grid, strongly coupling inter-grid relations that guarantee the conservation and accuracy properties of multiresolution finite volume schemes. The discrete Poisson equation is solved at once over the entire computational domain as a completely separate process. The accuracy and numerical performance of the method are assessed in the context of streamer discharge simulations.

1. Introduction

In numerous scientific applications one has to deal with the numerical solution of elliptic PDEs, like Poisson equations, coupled with evolutionary PDEs to address the numerical simulation of time-dependent physical processes. One major example is given, for instance, by the so-called projection methods [1,2], widely investigated, extended, and implemented in the literature to solve the incompressible Navier–Stokes equations (see, *e.g.*, [3] and references therein). Solving Poisson equations is also very common in plasma physics simulations. As an example, in the framework of a drift-diffusion model consisting of a set of continuity equations for charged species coupled with a Poisson equation for the electric potential, non-linear ionization waves also called streamers can be simulated [4,5]. In either situation Poisson-type equations must be solved (often several times) at every time-step throughout the numerical simulation, a task that depending on the size

* Corresponding author at: CCSE, Lawrence Berkeley National Laboratory, 1 Cyclotron Rd. MS 50A-1148, 94720 Berkeley, CA, USA.
E-mail addresses: MDGonzalez@lbl.gov (M. Duarte), zbona@physics.muni.cz (Z. Bonaventura), marc.massot@ecp.fr (M. Massot), anne.bourdon@lpp.polytechnique.fr (A. Bourdon).
[1] Present address: LPP, CNRS, Ecole Polytechnique, UPMC, Paris XI, 91128 Palaiseau, France.

http://dx.doi.org/10.1016/j.jcp.2015.02.038

and complexity of the problem can easily become cumbersome in both CPU time and memory. In particular phenomena characterized by propagating fronts, as considered in this work, commonly require a sufficiently fine spatial representation and potentially large systems of equations need then to be solved.

In this regard grid adaptation for time-dependent problems disclosing localized fronts is specifically designed to yield high data compression and hence important savings in computational costs (see, *e.g.*, [6,7]). Among the many adaptive meshing approaches developed in the literature, we consider in this work adaptive multiresolution schemes based on [8,9], namely the multiresolution finite volume scheme introduced in [10] for conservation laws. Besides the inherent advantages of grid adaptation, multiresolution techniques rely on biorthogonal wavelet decomposition [11] and thus offer a rigorous mathematical framework for adaptive meshing schemes [12,13]. In this way not only approximation errors coming from grid adaptation and thus data compression can be tracked, but general and robust techniques can be built since the wavelet decomposition is independent of any physical particularity of the problem and accounts only for the spatial regularity of the discrete variables at a given simulation time.

Adaptive multiresolution schemes have been successfully implemented for the simulation of compressible fluids modeled by Euler or Navier–Stokes equations (see, *e.g.*, [14–16] and references therein), as well as for the numerical solution of time-dependent parabolic [17,18] and stiff parabolic PDEs [19–21]. Nevertheless, to the best of our knowledge this is the first attempt to develop a Poisson solver in the context of the adaptive multiresolution finite volume method introduced in [10] for evolutionary problems. Previously, such a solver was introduced in [22] in the context of wavelet collocation methods for evolutionary PDEs developed in [23,24]. Analogous to multiresolution schemes, wavelet collocation methods assure adaptive meshing capabilities within a user-defined accuracy exploiting the mathematical properties of wavelet decomposition (for a recent review on wavelet methods see [25] and references therein). Notice that an important amount of research has been conducted in the past decades to solve elliptic PDEs using wavelet methods and multiresolution representations (see, for instance, [26–30]). In this context the numerical solution of an elliptic PDE is in general performed using compressed representations of the problem in an appropriate wavelet space, within a solid mathematical framework (see [31] and references therein). Here we do not consider wavelet methods to solve elliptic PDEs, but rather aim at developing a numerical strategy to discretize and solve Poisson equations on dynamically adapted finite volume grids generated by means of a multiresolution analysis.

Dynamic meshing techniques for finite volume discretizations are usually implemented by defining a set of embedded grids with different spatial resolution. Particular attention must be addressed to the inter-grid interfaces in order to consistently define the discrete operations there. Otherwise, potential mismatches may lead to substantial differences in the numerical approximations as well as loss of conservation (see [32] for a detailed discussion). The most common way of solving an elliptic PDE on this type of adapted grid consists in solving the discrete system level-wise, that is, considering one grid-level at a time followed by inter-level operations to synchronize shared interfaces at different grid-levels as well as overlapped regions. Computations are thus successively performed over partial regions at a uniform mesh resolution until the problem is entirely solved on the adapted grid. Some examples can be found, for instance, in [32–36]. For intensive computations iterative linear solvers based on geometric multigrid schemes are often implemented, taking advantage of the multi-mesh representation of the problem [32,33,35]. In particular the Poisson solver in [22] also implements a level-wise approach where a finite difference discretization is considered.

The main objective of this paper is to develop a Poisson solver on dynamically adapted grids generated with a multiresolution finite volume scheme. In particular we investigate the influence of data compression on the accuracy of approximations obtained with Poisson equations discretized on an adapted multiresolution mesh. One novelty of this paper in terms of elliptic solvers on adapted grids is that instead of solving the discrete equations level-wise throughout the set of embedded grids, we have conceived a numerical procedure to represent the elliptic operators discretized directly on the adapted grid, that is, on a mesh consisting of cells with different spatial resolution. The algorithm relies on a local reconstruction of uniform-grid zones at inter-level interfaces by means of multiresolution operations between consecutive grid-levels that guarantee the conservation and accuracy properties of multiresolution schemes. This approach results in a separate algebraic system completely independent of any consideration related to the adaptive meshing scheme or its corresponding data structure, as well as of the numerical integration of the time-dependent PDEs associated with the model. The resulting discrete system can thus be solved at once over the whole computational domain with no need of grid overlapping by considering an appropriate linear solver.

The performance of the strategy is assessed in the context of streamer discharge simulations at atmospheric pressure. The detailed physics of these discharges reveals an important time–space multi-scale character [37]. Grid adaptation is therefore highly desirable and was already considered, for instance, in [34,38,39]. In [40] we introduced a time–space adaptive numerical scheme with error control to simulate propagating streamers on multiresolution grids. Nevertheless, a simplified geometry was considered there in order to avoid the numerical solution of a multi-dimensional Poisson equation. The present work describes the required fundamentals and further developments needed to solve Poisson equations on a finite volume adapted grid according to the approach established in [40]. The latter aims at assuring a tracking capability of the numerical errors and a full resolution of the equations on the adapted grid.

The paper is organized as follows. We give in Section 2 a short introduction on multiresolution finite volume schemes and describe the data compression errors associated with Poisson equations discretized on multiresolution grids. In Section 3 we