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Positivity preserving high-order local discontinuous Galerkin method for parabolic equations with blow-up solutions

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ABSTRACT

In this paper, we apply positivity-preserving local discontinuous Galerkin (LDG) methods to solve parabolic equations with blow-up solutions. This model is commonly used in combustion problems. However, previous numerical methods are mainly based on a second order finite difference method. This is because the positivity-preserving property can hardly be satisfied for high-order ones, leading to incorrect blow-up time and blow-up sets. Recently, we have applied discontinuous Galerkin methods to linear hyperbolic equations involving δ-singularities and obtained good approximations. For nonlinear problems, some special limiters are constructed to capture the singularities precisely. We will continue this approach and study parabolic equations with blow-up solutions. We will construct special limiters to keep the positivity of the numerical approximations. Due to the Dirichlet boundary conditions, we have to modify the numerical fluxes and the limiters used in the schemes. Numerical experiments demonstrate that our schemes can capture the blow-up sets, and high-order approximations yield better numerical blow-up time.

1. Introduction

In this paper, we will develop and analyze the positivity-preserving high-order local discontinuous Galerkin methods for parabolic equations with reaction terms in one space dimension

$$\begin{aligned} &u_t = (u^\alpha)_{xx} + s(u), \quad x \in [x_a, x_b], \\ &u_0(x) = u(x,0) \geq 0, \quad x \in [x_a, x_b], \end{aligned} \tag{1}$$

as well as its two dimensional extension, where $\alpha \geq 1$ is a parameter and $s(u) \geq 0$. We consider Dirichlet boundary conditions

$$u(x_a, t) = g_1(t), \quad \text{and} \quad u(x_b, t) = g_2(t).$$

The source term $s(u)$ is chosen to be superlinear, i.e. $s(u) = u^m$ with $m \geq 1$. The initial condition $u = u_0(x)$ is assumed to be non-negative. By a maximum-principle, the exact solution $u \geq 0$ for $t \in (0, T)$. Here $(0, T)$ is the maximal time interval

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of existence of the solution u. The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite, the solution u develops a singularity in finite time, namely

$$\lim_{t\to T^-} \|u(\cdot,t)\|_\infty = \infty,$$

where $\|u\|_\infty$ is the standard L^∞-norm of u on $[x_a, x_b]$. In this case, we say that the solution u blows up in finite time and the time T is called the blow-up time of the solution u, and $\{x \in [x_a, x_b] | u(x,t) \to \infty \text{ as } T \to T^-\}$ is called the blow-up set. In this paper, we assume the boundary is not included in the blow-up set. Therefore, the exact solution at the boundary should be smaller than the values nearby.

We can regard (1) as a mathematical model of combustion, and the unknown variable u can be interpreted as the temperature. We can rewrite (1) as

$$\begin{aligned} &u_t = (a(u)u_x)_x + s(u), \quad x \in [x_a, x_b], \\ &u(x,0) \ge 0, \quad x \in [x_a, x_b], \end{aligned} \tag{2}$$

with $a(u) = \alpha u^{\alpha-1}$, which can be considered as a nonlinear heat conduction coefficient of the medium. If $\alpha = 1$, (2) reduces to the well-studied semilinear heat equation [4]. Such problems have been investigated by many authors and existence and uniqueness of a classical solution have been proved (see e.g. [26,34,6,5,20,21,29,25,41] for some recent works). Under some assumptions, it is also shown that the classical solution blows up in finite time and the blow-up time has been estimated. Moreover, many numerical methods are also been studied in [11,8,9,28,13,22,35,2,7,31,10,1,27]. However, to the best knowledge of the authors, all the previous methods are based on second-order finite difference methods, which guarantee the positivity of the numerical approximations automatically under some special requirements for the meshes. For high-order methods, strong oscillations near the singularities may send physically positive quantities negative, and negative coefficient of the diffusion term will make the numerical solution blow up immediately, leading to wrong blow-up time and blow-up sets. In this paper, we would like to apply high-order positivity-preserving local discontinuous Galerkin methods to such problems, and numerically study the blow-up time, blow-up locations and the behavior of the solutions before blow-up. We will also use numerical experiments to demonstrate the significance of the positivity-preserving technique.

The study of the blow-up solutions is important and challenging. Since the exact solutions may not be smooth or bounded at finite time, the numerical schemes might yield poor approximations. Recently, we [38] have applied discontinuous Galerkin (DG) methods to obtain good approximations for PDEs with δ-singularity, one special unbounded singularity. In [38], we proved the superconvergence results for linear hyperbolic equations with singular initial data and singular source terms. Moreover, several numerical examples were also given in [38–40] to demonstrate the advantages of the DG scheme in approximating δ-singularities for both linear and nonlinear hyperbolic equations. In this paper, we will continue this approach, and use the high-order DG methods to solve parabolic PDEs with blow-up solutions.

The DG method was first introduced in 1973 by Reed and Hill [30], in the framework of neutron linear transport. Later, the method was applied by Johnson and Pitkäranta to a scalar linear hyperbolic equation and the L^p-norm error estimate was proved [24]. Subsequently, Cockburn et al. developed Runge–Kutta discontinuous Galerkin (RKDG) methods for hyperbolic conservation laws in a series of papers [16,15,14,17]. In [18], Cockburn and Shu first introduced the LDG method to solve the convection–diffusion equation. Their idea was motivated by Bassi and Rebay [3], where the compressible Navier–Stokes equations were successfully solved.

The idea of the positivity-preserving technique we would like to apply in this paper is different from the one used before in [43–45]. In [45], the authors construct second-order positivity-preserving DG schemes and argued that it was impossible to construct a third-order positivity-preserving DG scheme following the same approach. Recently, in [36], the author applied the flux limiter and constructed maximum-principle-satisfying finite difference methods, which further generalized to high-order positivity-preserving DG methods for convection diffusion problems. The DG method applied in [36] is based on Ultra-weak DG [12]. We will consider LDG method and study the positivity-preserving technique. Another related work can be found in [42], where the positivity-preserving technique for porous medium equations have been analyzed. However, only a second-order scheme was considered and the technique can hardly be extended to high-order schemes. Another crucial problem is the boundary treatment for the LDG method. Numerical experiments demonstrate that the general LDG method will degenerate accuracy when solving problems with Dirichlet boundary conditions. Following [19], we would like to add a penalty term in the flux at the boundary. With the penalty term, the convergence rate is optimal. Moreover, we will also theoretically prove that the penalty term is required for the positivity-preserving technique.

The organization of this paper is as follows. In Section 2, we will present the positivity-preserving high-order LDG methods in one and two space dimensions, and the implementation of the Dirichlet boundary condition. Some numerical experiments will be given in Section 3. We will end in Section 4 with some concluding remarks and remarks on future work.

2. Positivity-preserving high-order local discontinuous Galerkin methods

In this section, we present the positivity-preserving technique applied to the LDG methods for parabolic equations subject to Dirichlet boundary conditions. We first introduce the LDG methods and discuss how to enforce the Dirichlet boundary conditions. Then we describe how to apply the positivity-preserving flux limiter to the numerical fluxes in the scheme.