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Diffusion approximations and domain decomposition method of linear transport equations: Asymptotics and numerics ☆

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ABSTRACT

In this paper we construct numerical schemes to approximate linear transport equations with slab geometry by diffusion equations. We treat both the case of pure diffusive scaling and the case where kinetic and diffusive scalings coexist. The diffusion equations and their data are derived from asymptotic and layer analysis which allows general scattering kernels and general data. We apply the half-space solver in [20] to resolve the boundary layer equation and obtain the boundary data for the diffusion equation. The algorithms are validated by numerical experiments and also by error analysis for the pure diffusive scaling case.

1. Introduction

Linear transport equations are widely used to model the interaction of particles with background media through various processes such as scattering, absorption, and emission. Many interesting physical systems exhibit heterogeneity that involve multiple temporal or spatial scales. In this paper we focus on efficient numerical simulations for linear transport equations which exhibit diffusive regime in part of or the whole domain. More precisely, with slab geometry, the particle density function f in our model depends on a one-dimensional spatial variable $x \in [a, b]$ and a one-dimensional angular variable $\mu \in [-1, 1]$. The transport equation has the reduced form

$$\begin{aligned} &\epsilon\, \partial_t f + \mu \partial_x f + \frac{\sigma(x)}{\epsilon} \mathcal{L} f = 0, \\ &f|_{x=a} = \phi_a(t, \mu), \qquad \mu > 0, \\ &f|_{x=b} = \phi_b(t, \mu), \qquad \mu < 0, \\ &f|_{t=0} = \phi_0(x, \mu), \end{aligned} \tag{1.1}$$

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where ϕ_a, ϕ_b are given incoming data at the boundary and ϕ_0 is the given initial data. The collision operator considered in this paper has the form

$$\mathcal{L}f = f - \int_{-1}^{1} \kappa(\mu, \mu') f(\mu') \, \mathrm{d}\mu', \tag{1.2}$$

where $\mathrm{d}\mu'$ is the regular Lebesgue measure. The scattering kernel κ satisfies that

$$\kappa \geq 0, \qquad \kappa(\mu, \mu') = \kappa(\mu', \mu), \qquad \int_{-1}^{1} \kappa(\mu, \mu') \, \mathrm{d}\mu' = 1 \quad \text{for all } \mu \in [-1, 1].$$

The parameter ϵ is the mean free path which is small compared with typical macroscopic length scale in the diffusion region. We will consider two cases for the coefficient $\sigma(x)$:

1) $\sigma(x) = 1$ for all $x \in [a, b]$ such that we have diffusive scaling over the whole domain;
2) the system contains two scales such that

$$\sigma(x) = \begin{cases} \epsilon, & x \in [a, x_m], \\ 1, & x \in [x_m, b], \end{cases}$$

for some $x_m \in (a, b)$, such that we have a kinetic scaling on the left domain and a diffusive scaling on the right. These two regions are coupled together at the interface $x = x_m$.

We note that while we focus on system (1.1) which is one-dimensional in both the spatial and the angular variables, it is possible to extend to higher dimensional systems with simple geometry. Particular examples include the upper half-space $(\mathbb{R}^n)^+$ in the pure diffusion case and flat interface in the kinetic–fluid coupling case. We also comment that our method can be applied to general linear or linearized kinetic equations such as the linearized Boltzmann equations.

In this paper we aim at designing efficient multiscale algorithms for (1.1) based on asymptotic analysis and domain decomposition. It is well known that direct simulations of the transport equation in the diffusion region are usually rather expensive thus unfavorable. On the other hand, diffusion equations with proper data can provide good approximations to the kinetic equation when ϵ is small (see for example [12,13]). We will follow the latter route and use diffusion approximations wherever applicable. The main difficulty of this method lies in obtaining accurate matching boundary and initial conditions for the diffusion equation. The easier part is to obtain the initial data: at the leading order, it can be derived by directly projecting the given kinetic initial state onto the null space of the scattering operator $\mathcal{L}$.

Finding the boundary data on the other hand is more involved both asymptotically and numerically. For given kinetic incoming data, one can show by formal asymptotic analysis that the matching boundary data of the diffusion equation is determined by the end-state of the solution to a half-space equation. Therefore, accurate solvers of half-space equations will provide crucial tools for our approximation. Having this in mind, we developed a numerical method in [20] that can efficiently solve the half-space equations. In the current paper, we will apply this half-space solver to obtain the boundary data for the diffusion equations numerically.

In summary, in the numerical scheme for the pure diffusion case, we will resolve the boundary layer equations at the two endpoints $x = a, b$ to retrieve boundary data for the diffusion equation and use the projected kinetic initial data as its initial condition. We will compare thus-obtained diffusion solution to the solution of kinetic equation and show convergence rates in terms of ϵ. We will also derive some formal error estimates in the L^2-spaces. The error analysis follows the classical methodology of constructing approximate solution that involves all the layers [4]. Since we are studying time-dependent case, normally there will be three types of layers involved: boundary, initial, and initial–boundary layers. The major assumption that we make here is to assume that the initial–boundary layer equation is well-posed and its solution decays at least as fast as the reciprocal of time. The initial and boundary layers on the other hand can be shown to have an exponential decay. We will treat the general cases where the derived data for the diffusion equation are allowed to be incompatible so that the derivatives of the heat solution can be unbounded.

In the formal asymptotic analysis of the kinetic-diffusion coupling case with general initial data, there are boundary, initial, and initial–boundary layers that form at the interface in the diffusion region. Solutions to these layers will have influence on the kinetic part. Our numerical scheme, however, only takes the boundary-layer feedback into account and ignores the other feedbacks from the initial and initial–boundary layers at the interface. This way we can decouple the kinetic and diffusion parts at the leading order. This decoupling idea is a feature of the domain-decomposition method developed in [13]. In particular, at the leading order, the kinetic part satisfies a closed system whose boundary condition at the interface is given by the Albedo operator defined in (2.18). By this we can fully solve the leading order decoupled kinetic equation in the kinetic region. Using the solution from the kinetic regime at the interface as the given incoming data, we then approximate the kinetic equation in the diffusion region by the diffusion equation via the same scheme for the pure diffusion case. We comment that although we do not have rigorous analysis for estimating the errors induced by ignoring