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A multi-dimensional high-order discontinuous Galerkin method based on gas kinetic theory for viscous flow computations

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ABSTRACT

This paper presents a high-order discontinuous Galerkin (DG) method based on a multi-dimensional gas kinetic evolution model for viscous flow computations. Generally, the DG methods for equations with higher order derivatives must transform the equations into a first order system in order to avoid the so-called "non-conforming problem". In the traditional DG framework, the inviscid and viscous fluxes are numerically treated differently. Differently from the traditional DG approaches, the current method adopts a kinetic evolution model for both inviscid and viscous flux evaluations uniformly. By using a multi-dimensional gas kinetic formulation, we can obtain a spatial and temporal dependent gas distribution function for the flux integration inside the cell and at the cell interface, which is distinguishable from the Gaussian Quadrature point flux evaluation in the traditional DG method. Besides the initial higher order non-equilibrium states inside each control volume, a Linear Least Square (LLS) method is used for the reconstruction of smooth distributions of macroscopic flow variables around each cell interface in order to construct the corresponding equilibrium state. Instead of separating the space and time integrations and using the multistage Runge–Kutta time stepping method for time accuracy, the current method integrates the flux function in space and time analytically, which subsequently saves the computational time. Many test cases in two and three dimensions, which include high Mach number compressible viscous and heat conducting flows and the low speed high Reynolds number laminar flows, are presented to demonstrate the performance of the current scheme.

1. Introduction

In order to improve the reliability of numerical methods and present accurate flow computations, the development of high order (>2nd) schemes has been under intensive investigation recently. Most finite volume (FV) methods are based on the piecewise constant representation of flow variables and resort the reconstruction techniques to obtain high order

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accuracy in space. Generally these methods are effective for structured meshes, but may face reconstruction problem on arbitrary grids in the multidimensional unstructured mesh cases due to the use of extended stencils. In order to avoid difficulties in the reconstruction, one possible way is to develop compact methods, and the discontinuous Galerkin (DG) method [1] becomes one of these idealized choices. For the DG method, the higher accuracy is achieved by means of higher order polynomial approximation inside each cell. Only the information from adjacent cells with common cell interfaces is needed for the update of the degree of freedom of the cell. Therefore, the DG method can deliver higher order accurate solutions without solely relying on the reconstruction techniques and large stencils.

The DG method was firstly introduced by Reed and Hill [2] and applied to a linear transport equation by Lesaint and Raviart [3]. Chavent and Salzano [4] firstly adapted the method to a nonlinear case. Cockburn and Shu [1,5] further developed the method in a series of papers, in which a framework to solve the nonlinear time dependent hyperbolic conservation laws was established. For convection–diffusion equations, the DG method proposed by Cockburn and Shu [1,5] cannot be directly applied, because the discontinuities appearing at the cell interface are not regular enough to handle higher order derivatives [6]. The alternative formulation proposed by Bassi and Rebay [7] is to first transform the convection–diffusion equations into a first order system. This technique successfully extends the DG method to the Navier–Stokes (NS) flow computation. Cockburn and Shu [8] proposed a local discontinuous Galerkin method based on a similar formulation.

Alternatively, the Navier–Stokes solutions can be recovered using a gas-kinetic formulation [9–13], where a kinetic flux function including both inviscid and viscous terms can be obtained in the kinetic evolution model. In the gas kinetic scheme, the fluxes are constructed based on the integral solution of the gas kinetic Bhatnagar–Gross–Krook (BGK) model, which presents a multiscale evolution process from a non-equilibrium to an equilibrium state, with the inclusion of time evolution of both flow variables and their derivatives. Different from the Riemann problem, the flow dynamics from a higher order initial reconstruction is explicitly followed. Xu [12] firstly proposed a one dimensional DG method by using a 2nd order BGK scheme for the flux computation. Liu and Xu [9] adopted the 2nd order BGK scheme on each Gaussian Quadrature point, where the directional splitting method is used for the 2D cases. Ni et al. [11] and Luo et al. [10] also used the 2nd order BGK scheme for the flux evaluation. In this paper, we will use a multi-dimensional 3rd order gas-kinetic BGK scheme, which is similar with the finite volume BGK scheme [14], to evaluate the fluxes both inside a cell and at a cell interface. Based on the multi-dimensional gas kinetic formulation, a spatial and temporal dependent gas distribution function can be evaluated explicitly, which can be integrated analytically in both space and time without using the integration formula based on the Gaussian Quadrature points. In the current approach, an LLS method is used for the equilibrium state construction around a cell interface, which is approached by the initial non-equilibrium state in the time evolution process.

The paper is organized as follows. Section 2 is the construction of the numerical method, which is composed of the gas kinetic discontinuous Galerkin scheme, the gas distribution function construction, and an LLS method for the equilibrium state construction. Section 3 shows some numerical examples and the results. The last section draws the conclusion.

2. Numerical method

2.1. Gas kinetic discontinuous Galerkin scheme

A 3D gas-kinetic BGK model is

$$\frac{\partial f}{\partial t} + \vec{u} \bullet \nabla f = \frac{g - f}{\tau}, \tag{1}$$

where $\vec{u} = (u, v, w)$ is the particle velocity vector, f is the gas distribution function, g is the equilibrium state approached by f, ∇f is the gradient of f with respect to $\boldsymbol{x} = (x, y, z)$, τ is the particle collision time which is related to the viscosity and heat conduction coefficients, and t is the physical time.

The equilibrium state is a Maxwellian distribution,

$$g = \rho\left(\frac{\lambda}{\pi}\right)^{\frac{K+3}{2}} e^{-\lambda[(u-U)^2+(v-V)^2+(w-W)^2+\xi^2]}, \tag{2}$$

in which ρ is the density, U, V and W are the macroscopic velocity in x-, y- and z-direction, and λ is related to the gas temperature T by $\lambda = m/2kT$, where m is the molecular mass and k is the Boltzmann constant. The total number of degree of freedom K in the internal variable ξ is equal to $(5 - 3\gamma)/(\gamma - 1)$ and γ is the specific heat ratio. In the above equilibrium state g, the internal variable ξ^2 is equal to $\xi^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_K^2$. The relation between the macroscopic conservative variables and the distribution function is

$$\boldsymbol{Q} = (\rho, \rho U, \rho V, \rho W, \rho E)^{\mathrm{T}} = \int \boldsymbol{\psi} f \mathrm{d}\Xi = \int \boldsymbol{\psi} g \mathrm{d}\Xi, \tag{3}$$

where the vector of moments $\boldsymbol{\psi}$ is

$$\boldsymbol{\psi} = \left(1, u, v, w, \frac{1}{2}\left(u^2 + v^2 + w^2 + \xi^2\right)\right)^{\mathrm{T}}, \tag{4}$$

and $\mathrm{d}\Xi = \mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}\xi_1\mathrm{d}\xi_2 \cdots \mathrm{d}\xi_K$ is the volume element in the phase space.