Journal of Computational Physics 274 (2014) 191–207

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

The solution of the scalar wave equation in the exterior of a sphere

Leslie Greengard [a,*,1], Thomas Hagstrom [b,2], Shidong Jiang [c,3]

[a] Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, United States
[b] Department of Mathematics, Southern Methodist University, PO Box 750156, Dallas, TX 75275, United States
[c] Department of Mathematical Sciences, New Jersey Institute of Technology, Newark, NJ 07102, United States

ARTICLE INFO

Article history:
Received 3 August 2013
Received in revised form 22 April 2014
Accepted 22 May 2014
Available online 2 June 2014

Keywords:
Wave equation
Scattering
Time-domain
Separation of variables
Numerical stability

ABSTRACT

We derive new, explicit representations for the solution to the scalar wave equation in the exterior of a sphere, subject to either Dirichlet or Robin boundary conditions. Our formula leads to a stable and high-order numerical scheme that permits the evaluation of the solution at an arbitrary target, without the use of a spatial grid and without numerical dispersion error. In the process, we correct some errors in the analytic literature concerning the asymptotic behavior of the logarithmic derivative of the spherical modified Hankel function. We illustrate the performance of the method with several numerical examples.

© 2014 Elsevier Inc. All rights reserved.

1. Introduction

In this paper, we consider a simple problem, namely the solution of the scalar wave equation

$$u_{tt} = \Delta u, \quad t > 0, \tag{1}$$

subject to homogeneous initial conditions

$$u(r,\theta,\phi,0) = 0, \qquad u_t(r,\theta,\phi,0) = 0 \tag{2}$$

in the exterior of the unit sphere. Here, (r,θ,ϕ) denote the spherical coordinates of a point in $\mathbb{R}^3$ with $r > 1$. Standard textbooks on mathematical physics (such as [5,11]) present exact solutions for the time-harmonic cases governed by the Helmholtz equation, but generally fail to discuss the difficulties associated with the fully time-dependent case (1). As we shall see, it is a nontrivial matter to develop closed-form solutions, and a surprisingly subtle matter to develop solutions that can be computed without catastrophic cancellation.

* Corresponding author.
E-mail addresses: greengard@courant.nyu.edu (L. Greengard), thagstrom@smu.edu (T. Hagstrom), shidong.jiang@njit.edu (S. Jiang).

[1] Supported in part by the Office of the Assistant Secretary of Defense for Research and Engineering and AFOSR under NSSEFF Program Award FA9550-10-1-0180 and in part by the U.S. Department of Energy under Contract DEFG0288ER25053.
[2] Supported in part by the Army Research Office under agreement W911NF-09-1-0344 and by the NSF under Grant OCI-0904773.
[3] Supported in part by NSF under Grant CCF-0905395.

http://dx.doi.org/10.1016/j.jcp.2014.05.031
0021-9991/© 2014 Elsevier Inc. All rights reserved.

In this paper, we restrict our attention to boundary value problems with Dirichlet or Robin conditions. We consider the Dirichlet problem first, and assume we are given data on the boundary of the unit sphere of the form:

$$u(1,\theta,\phi,t) = f(\theta,\phi,t). \tag{3}$$

It is natural to begin by expanding both u and f in terms of spherical harmonics.

$$\begin{aligned} u(r,\theta,\phi,t) &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} u_{nm}(r,t)Y_{nm}(\theta,\phi), \\ f(\theta,\phi,t) &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{nm}(t)Y_{nm}(\theta,\phi), \end{aligned} \tag{4}$$

where

$$Y_n^m(\theta,\phi) = \sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\,P_n^{|m|}(\cos\theta)e^{im\phi}\,, \tag{5}$$

$P_n(x)$ is the standard Legendre polynomial of degree n, and the associated Legendre functions P_n^m are defined by the Rodrigues' formula

$$P_n^m(x) = (-1)^m\big(1-x^2\big)^{m/2}\frac{d^m}{dx^m}P_n(x).$$

We let $\hat{u}_{nm}(r,s)$ and $\hat{f}_{nm}(s)$ denote the Laplace transforms of $u_{nm}(r,t)$ and $f_{nm}(t)$:

$$\hat{u}_{nm}(r,s) = \int_0^\infty e^{-st}u_{nm}(r,t)dt, \tag{6}$$

$$\hat{f}_{nm}(s) = \int_0^\infty e^{-st}f_{nm}(t)dt. \tag{7}$$

It is straightforward [1] to see that $\hat{u}_{nm}(r,s)$ satisfies the linear second order ordinary differential equation (ODE)

$$r^2\hat{u}_{nm}(r,s)_{rr} + 2r\hat{u}_{nm}(r,s)_r - \big[s^2r^2 + n(n+1)\big]\hat{u}_{nm}(r,s) = 0,$$

for which the decaying solution as $r\to\infty$ is the modified spherical Hankel function $k_n(sr)$. It follows that

$$\hat{u}_{nm}(r,s) = c_{nm}(s)k_n(sr).$$

Matching boundary data on the unit sphere, we have $c_{nm}(s) = \hat{f}_{nm}(s)/k_n(s)$, and

$$\hat{u}_{nm}(r,s) = \frac{k_n(sr)}{k_n(s)}\hat{f}_{nm}(s). \tag{8}$$

The remaining difficulty is that we have an explicit solution in the Laplace transform domain, but we seek the solution in the time domain. For this, we write the right hand side of (8) in a form for which the inverse Laplace transform can carried out analytically. First, from [1,10,12], we have

$$k_n(z) = \frac{p_n(z)}{z^{n+1}}e^{-z} = \frac{\prod_{j=1}^n(z-\alpha_{n,j})}{z^{n+1}}e^{-z}, \tag{9}$$

where $\alpha_{n,j}$ $(j=1,\dots,n)$ are the simple roots of k_n lying on the open left half of the complex plane (see Fig. 5 for a plot of the zeros of k_{10} and k_{11}). Thus,

$$\begin{aligned} \frac{k_n(sr)}{k_n(s)} &= \frac{1}{r}e^{-s(r-1)}\prod_{j=1}^n\frac{s-\frac{1}{r}\alpha_{n,j}}{s-\alpha_{n,j}} \\ &= \frac{1}{r}e^{-s(r-1)}\left(1+\sum_{j=1}^n\frac{a_{n,j}(r)}{s-\alpha_{n,j}}\right), \end{aligned} \tag{10}$$

where the second equality follows from an expansion using partial fractions and the coefficients $a_{n,j}$ are given from the residue theorem by the formula:

Download English Version:

https://daneshyari.com/en/article/519903

Download Persian Version:

https://daneshyari.com/article/519903

Daneshyari.com