Journal of Computational Physics 274 (2014) 268–282

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A compact difference scheme for a two dimensional fractional Klein–Gordon equation with Neumann boundary conditions ☆

Seakweng Vong, Zhibo Wang *

Department of Mathematics, University of Macau, Av. Padre Tomás Pereira Taipa, Macau, China

ARTICLE INFO

Article history:
Received 16 December 2013
Received in revised form 19 May 2014
Accepted 13 June 2014
Available online 19 June 2014

Keywords:
Two dimensional fractional Klein–Gordon equation
Compact difference scheme
Stability
Convergence

ABSTRACT

In this paper, a high order finite difference scheme for a two dimensional fractional Klein–Gordon equation subject to Neumann boundary conditions is proposed. The difficulty induced by the nonlinear term and the Neumann conditions is carefully handled in the proposed scheme. The stability and convergence of the finite difference scheme are analyzed using the matrix form of the scheme. Numerical examples are provided to demonstrate the theoretical results.

1. Introduction

Klein–Gordon equations have wide applications in quantum mechanics and condensed matter physics. Interested readers can refer to [1–3] for practical applications of Klein–Gordon equations in superconductors, motion of pendule and dislocations in crystals.

In this paper, we study Klein–Gordon equations with time fractional derivatives. Fractional derivatives are generalizations of classical ones. We remark that the generalization is not just for pure mathematical purpose. Actually, it is found that sub-diffusion process can be described more accurately by using fractional derivatives. The books [4,5] give theoretical analysis for fractional differential equations, while our study concentrates on their numerical methods. Since fractional derivatives have historical dependence, this makes deriving numerical schemes for fractional differential equations a challenging problem.

We review briefly here some works that are related to our current study. In [6], Sun and Wu constructed a difference scheme to solve the fractional diffusion-wave equation, which has been proved to be unconditionally stable. Recently, Zhuang et al. [7] investigated the stability and convergence of an implicit numerical method for the anomalous sub-diffusion equation by the energy method. A compact finite difference scheme for this equation was then presented by Cui in [8], where the local truncation error and the stability were studied by the Fourier method. By a transformation of this problem, Gao and Sun [9] proposed a high order scheme to improve the temporal convergence order. Lately, based on [9], Ren et al. established two compact schemes for fractional diffusion equations with Neumann boundary conditions in [10,11]. A high order compact difference scheme for the fractional Cattaneo equation was derived in [12]. In the past few years, Dehghan

☆ This research is supported by the Macao Science and Technology Development Fund FDCT/001/2013/A and the grant MYRG086(Y2-L2)-FST12-VSW from University of Macau.

* Corresponding author.
E-mail addresses: swvong@umac.mo (S. Vong), zhibowangok@gmail.com (Z. Wang).

http://dx.doi.org/10.1016/j.jcp.2014.06.022
0021-9991/

et al. have established many results in this area [13–20]. Especially, in [20], they proposed high order difference schemes for fractional Cattaneo equations, linear fractional Klein–Gordon and dissipative Klein–Gordon equations. That paper and the references therein provided some recent progress related to our current study. By interpolating polynomials, Sousa and Li [21,22] constructed two implicit schemes for equations with the Riemann–Liouville fractional derivative and the Caputo fractional derivative respectively, which are of second order accuracy, while in [23], high order finite difference schemes based on the weighted and shifted Grünwald difference operator were developed for solving space fractional diffusion equations. Along with these studies, we have proposed several high order finite difference schemes to solve time fractional differential equations with Caputo fractional derivative [24–26].

Most of the studies mentioned above concern with problems without nonlinear interferences. Nonlinear terms in fractional differential equations make the study more difficult. Recently some works on numerical methods for nonlinear fractional differential equations have been done [27–37]. In [27], Li et al. proposed a Galerkin finite element method for nonlinear time–space fractional sub-diffusion and super-diffusion equations. By using the homotopy analysis method, in [28], the authors succeeded in solving the nonlinear fractional differential equations with high accuracy and efficiency. A modified anomalous time fractional sub-diffusion equation with a nonlinear source term was studied in [29,30], where a finite difference scheme of first order temporal accuracy and fourth order spatial accuracy was proposed. In [31], Cui constructed a fourth-order compact scheme for the one-dimensional Sine–Gordon equation. The resulting fully discrete nonlinear finite difference equation was solved by a predictor–corrector scheme. In [32], a difference scheme was derived for coupled nonlinear Schrödinger equations with the Riesz spatial fractional derivatives. A spatially second-order scheme for a nonlinear fractional Bloch–Torrey equation was recently studied in [33]. Wang and his colleagues have established many results on nonlinear partial differential equations [34–37].

The objective of this paper is to study high order finite difference schemes for the following two dimensional nonlinear fractional Klein–Gordon equation

$$ {}_0^C D_t^\alpha u - \Delta u + u^3 = f, \qquad \mathbf{x} \in \Omega = (0, L_1) \times (0, L_2), \quad 0 < t \le T, \tag{1} $$

subject to the initial conditions

$$ u(\mathbf{x}, 0) = \phi(\mathbf{x}), \qquad \frac{\partial u(\mathbf{x}, 0)}{\partial t} = \psi(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega} = \Omega \cup \partial\Omega, \tag{2} $$

and the boundary conditions

$$ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}, t) = 0, \qquad \mathbf{x} \in \partial\Omega, \quad 0 < t \le T, \tag{3} $$

where

$$ {}_0^C D_t^\alpha u = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(\mathbf{x}, s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}} $$

is the Caputo fractional derivative of order $\alpha \in (1, 2)$ with $\Gamma(\cdot)$ being the gamma function.

Very recently, inspired by results in [34,35], we have established a compact scheme of order $\tau^{3-\alpha} + h^4$ for the one dimensional fractional Klein–Gordon equation with Dirichlet boundary conditions, where τ and h are the time and space step sizes respectively. However, a critical lemma used in [34,35] is not readily to be obtained when the Dirichlet boundary conditions are replaced by the Neumann boundary conditions (3). In this paper, by a transformation of the matrix form of our proposed scheme, we succeed in getting the desired estimate (Lemma 3.3). Another difficulty for dealing with the Neumann boundary conditions is that, due to the nonlinear term u^3, an extra term of order h^3 is induced when the compact operator is acted on the boundary. In order to match the accuracy with that at interior grid points, this extra term is included in the proposed scheme as an artificial nonlinearity. Finally, it is well known that L_∞ norm bound is not easy to get for problems with dimension greater than one. However, we have used this kind of bound in our previous study. To resolve this problem, a lemma in [36] is employed as a remedy. With all these efforts, we succeed in showing that our proposed scheme converges in $\|\cdot\|_H$ (defined as in Theorem 3.1) of order $\tau^{3-\alpha} + h_1^4 + h_2^4$.

This paper is organized as follows. A high order compact scheme is proposed in the next section. The stability and convergence of the compact scheme are analyzed in Section 3. In Section 4, numerical experiments are carried out to justify the theoretical results. The article ends with a brief conclusion.

2. The proposed compact finite difference scheme

To propose a compact scheme for (1)–(3), we let $h_1 = \frac{L_1}{M_1}$, $h_2 = \frac{L_2}{M_2}$ and $\tau = \frac{T}{N}$ be the spatial and temporal step sizes respectively, where M_1, M_2 and N are some given integers. For $i = 0, 1, \ldots, M_1$, $j = 0, 1, \ldots, M_2$ and $k = 0, 1, \ldots, N$, denote $x_i = ih_1$, $y_j = jh_2$, $t_k = k\tau$. For a grid function $u = \{u_{ij}^k | 0 \le i \le M_1,\ 0 \le j \le M_2,\ 0 \le k \le N\}$, we introduce the following notations: