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Finite element three-dimensional Stokes ice sheet dynamics model with enhanced local mass conservation ☆

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ABSTRACT

Parallel finite element nonlinear Stokes models have been successfully used for three-dimensional ice-sheet and glacier simulations due to their accuracy and efficiency, and their capability for easily handling highly irregular domains and different types of boundary conditions. In particular, the well-known Taylor–Hood element pair (continuous piecewise quadratic elements for velocity and continuous piecewise linear elements for pressure) results in highly accuracy velocity and pressure approximations. However, the Taylor–Hood element suffers from poor mass conservation which can lead to significant numerical mass balance errors for long-time simulations. In this paper, we develop and investigate a new finite element Stokes ice sheet dynamics model that enforces local element-wise mass conservation by enriching the pressure finite element space by adding the discontinuous piecewise constant pressure space to the Taylor–Hood pressure space. Through various numerical tests based on manufactured solutions, benchmark test problems, and the realistic Greenland ice-sheet, we demonstrate that, for ice-sheet modeling, the enriched Taylor–Hood finite element model remains highly accurate and efficient, and is physically more reliable and robust compared to the classic Taylor–Hood finite element model.

1. Introduction

Finite element methods have been successfully employed to numerically model and simulate 3D nonlinear Stokes ice sheet and glacier flows [1] because they have the flexibility to use unstructured grids that naturally conform to the rough terrain of the ice sheets and because such methods are based on variational formulations witch enables the natural application and implementation of diverse boundary conditions. The Taylor–Hood finite element pair is very popular for the discretization of Stokes equations in many settings [12] and thus is also popular for approximating the ice-sheet dynamics.

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This element pair consists of continuous piecewise quadratic elements for the velocity approximation and piecewise linear elements for the pressure; e.g., we have the Q2/Q1 finite element pair on hexahedral grids [17,8,9] and the P2/P1 finite element pair on tetrahedral grids [31,14,16]. Stability and error estimates for the Taylor–Hood element pair for linear Stokes problems were rigorously obtained in, e.g., [4,5] based on the well-known inf-sup condition. In particular, this method is of high-order accuracy, with approximations of the velocity and pressure being third-order and second-order accurate, respectively. For the nonlinear Stokes ice-sheet simulation, similar numerical accuracies still hold for such element as demonstrated in [14]. On the other hand, because the classic Taylor–Hood finite element pair on tetrahedra uses a continuous pressure space, the incompressibility constraint is only satisfied globally, e.g., the mass is only conserved exactly on the whole domain but not at the element level. For this reason, other finite element pairs for Stokes discretization have been proposed for which the pressure approximations are discontinuous across element boundaries. One such element pair enriches the pressure approximating space with the necessarily discontinuous piecewise constant finite element space, thus adding a constant-valued variable for each element to the classical Taylor–Hood element. As a result, mass conservation is guaranteed at the element level in the sense that the average of the divergence of the velocity over each element is zero. This enriched Taylor–Hood finite element approximation was first used by [10,11] on linear Stokes and Navier–Stokes problems and its stability was proven by [23,25,29] on triangular and rectangular meshes. The stability for the enriched Taylor–Hood element of even higher order accuracy on general meshes is proven in [6].

Based on our previous work on high-order finite element approximations on tetrahedral grids for simulating ice-sheet flows via the nonlinear three-dimensional Stokes model [14,16], in this paper we develop and test a new finite element Stokes ice-sheet dynamics model that uses the enriched Taylor–Hood finite element pair. Through comparisons of the simulation results between the two finite element models on some typical problems, we demonstrate that the new computational finite element model conserves mass almost perfectly and is also physically more reliable and robust.

The rest of the paper is organized as follows. In Section 2, we review the nonlinear Stokes mathematical model for ice-sheet dynamics. Then, in Section 3, we present the enriched Taylor–Hood finite element computational model for ice-sheet dynamics and discuss nonlinear and linear solution techniques. In Section 4 we present the results of several computational demonstrations, including some dealing with manufactured solutions, others with standard test problems in the literature, and further ones for the realistic Greenland ice-sheet geometry.

2. The nonlinear Stokes ice sheet dynamics

2.1. Governing equations

Let the 3D spatial domain Ω_t occupied by the ice sheet at a time $t \in [0, t_{max}]$ be defined as

$$\Omega_t = \left\{(x, y, z) \,\middle|\, z_b(x, y) \le z \le z_s(x, y, t) \text{ for } (x, y) \in \Omega_H\right\},$$

where Ω_H denotes the horizontal extent of the ice sheet, $z_s(x, y, t)$ defines the elevation of the top surface Γ_s of the ice sheet, and $z_b(x, y)$ defines the fixed bottom surface Γ_b of the ice sheet. In general, $z_b(x, y) \ne z_s(x, y, t)$ along the boundary of Ω_H, i.e., the margin of the ice sheet consists of a vertical cliff; thus the ice sheet also has a lateral boundary Γ_l. The dynamical behavior of ice sheets is mathematically modeled by the Stokes equations for the flow of an incompressible viscous fluid in the low Reynolds-number regime. Because the time scale of variations in the velocity and pressure fields is large, the entire material derivative is neglected. Furthermore, a nonlinear rheology, i.e., a nonlinear constitutive law is assumed. Letting $[0, t_{max}]$ denote the time interval of interest, we then have

$$\nabla \cdot \sigma + \rho \mathbf{g} = 0 \quad \text{in } \Omega_t \times [0, t_{max}], \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega_t \times [0, t_{max}], \tag{2}$$

where $\mathbf{u} = (u, v, w)^T$ denotes the velocity, σ the full stress tensor, ρ the density of ice, and $\mathbf{g} = (0, 0, -g)$ the gravitational acceleration. The full stress tensor σ can be decomposed in terms of the deviatoric stress τ and the static pressure p as

$$\sigma = \tau - p\mathbf{I} \quad \text{or} \quad \sigma_{ij} = \tau_{ij} - p\delta_{ij}, \tag{3}$$

where $p = -\frac{1}{3}tr(\sigma)$, δ_{ij} denotes the Kronecker delta tensor, and $\mathbf{I}$ the unit tensor. Combining (1) and (3), we obtain the *instantaneous* momentum balance equation

$$-\nabla \cdot \tau + \nabla p = \rho \mathbf{g} \quad \text{in } \Omega_t \times [0, t_{max}]. \tag{4}$$

The strain-rate tensor $\dot{\varepsilon}_{\mathbf{u}}$ is defined as

$$(\dot{\varepsilon}_{\mathbf{u}})_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{5}$$

The constitutive law for ice relates the deviatoric stress tensor τ to the strain-rate tensor $\dot{\varepsilon}_{\mathbf{u}}$ by the generalized Glen's flow law [19,21] based on the assumption of small strain-induced deformations:

$$\tau = 2\eta_{\mathbf{u}}\dot{\varepsilon}_{\mathbf{u}} \tag{6}$$