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Mass, momentum and energy conserving (MaMEC) discretizations on general grids for the compressible Euler and shallow water equations

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ABSTRACT

The paper explains a method by which discretizations of the continuity and momentum equations can be designed, such that they can be combined with an equation of state into a discrete energy equation. The resulting 'MaMEC' discretizations conserve mass, momentum as well as energy, although no explicit conservation law for the total energy is present. Essential ingredients are (i) discrete convection that leaves the discrete energy invariant, and (ii) discrete consistency between the thermodynamic terms. Of particular relevance is the way in which finite volume fluxes are related to nodal values. The method is an extension of existing methods based on skew-symmetry of discrete operators, because it allows arbitrary equations of state and a larger class of grids than earlier methods.

The method is first illustrated with a one-dimensional example on a highly stretched staggered grid, in which the MaMEC method calculates qualitatively correct results and a non-skew-symmetric finite volume method becomes unstable. A further example is a two-dimensional shallow water calculation on a rectilinear grid as well as on an unstructured grid. The conservation of mass, momentum and energy is checked, and losses are found negligible up to machine accuracy.

1. Introduction

Flow equations are considered consisting of a continuity and a momentum equation – expressing conservation of mass and momentum – plus a thermodynamic equation of state. No explicit equation for the evolution of total energy is present, but the latter can be derived analytically by combining the given equations. In absence of dissipative effects, total energy is then conserved, with an exchange between kinetic energy and thermodynamic potential energy. In the paper, discretizations of the flow equations will be discussed that similarly can be combined into a discrete conservation equation for total energy. The discussion will be held mainly for staggered computational grids, but it can also be applied to collocated grid arrangements. It is also extendable to formulations that include an internal energy equation, and will then likewise result in discrete conservation of total energy.

In this context, 'conservation' of a total quantity (like total mass, total momentum or total energy) does not mean that the total quantity does not change at all. The term 'conservation' is used to indicate that the total quantity changes *only* as a result of terms that change the quantity in the continuous model, of which the discretization is an approximation. Moreover, terms which change the total quantity only in a certain way in the continuous model (for instance, only create an increase or a decrease), should have the same property in the discretization. Hence, the discretization does not introduce any changes in the total amounts; only the appropriate physical terms do.

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In certain simulations, the conservation properties of the flow are very important for the correctness of the results. For instance, it is considered crucial in direct numerical simulation of turbulence (DNS), because the balance between energy dissipation and energy input determines the characteristics of the turbulent flow [5,18,31,36–38]. In shallow water simulations, both the momentum and energy balance are important in parts of the simulation domain where rapid fluctuations occur. In [26] a method is proposed that switches from a momentum conserving discretization to an energy conserving discretization, depending on the local flow characteristics. The method presented in the current paper is practical because the choice between momentum conservation and energy conservation is avoided: both are satisfied simultaneously.

In the papers mentioned thus far, the flow equations were mainly solved in a velocity-pressure formulation, either in conservative (divergence) form or in skew-symmetric form. An alternative is to formulate them in a rotational (vorticity) form. Especially in the literature regarding weather forecast, starting with Arakawa's seminal paper [1] (see also [16]), or shallow water equations, e.g. [2,24], this approach is followed. In this way it is possible to conserve energy in combination with enstrophy or helicity, e.g. [15], but the conservation of momentum is lost. This often is a serious drawback of the approach; see also the trade-off discussions on this issue in [18,19]. In a similar vain, in Hamiltonian particle-mesh methods the conservation of vorticity is advocated [6,9]. A comparison with such an approach would be interesting, as it is not yet known which quantity is the 'most relevant' to be preserved.

In many cases, finite volume methods are an effective way to construct conservative discretizations. However, there are two reasons why it is difficult to construct a fully conservative discretization method on a general (non-Cartesian) staggered grid. The main problem is that finite volume methods may be used to discretize the continuity and momentum equations, but in general it is not possible to combine the discrete continuity and momentum equations into a discrete energy equation. Already several decades ago, for incompressible flow [22] as well as compressible flow [7], it has been noted that the key ingredient for the construction of an energy-conserving discretization of the continuity and momentum equations is the preservation of the (skew)-symmetry properties of the differential operators in their discrete approximations. If these properties can be preserved, good results are obtained [8,10,14,31–33,36–38]. Also the related summation-by-parts schemes as introduced in [27] and the cosymmetry-preserving approach in [12] are designed to achieve this.

The other problem is that in a non-Cartesian staggered grid it is hard to conserve momentum with a finite volume method, because the discrete flow components each have a different direction. Yet, several attempts have been made in the literature to tackle the problem. A general framework is formed by the mimetic (or discrete calculus) methods developed at Los Alamos National Laboratory, e.g. [3,25]. Applications to unstructured staggered mesh schemes have been presented in [19–21,29].

The paper provides general guidelines for the construction of mass, momentum and energy conserving ('MaMEC') discretization schemes, also applicable to unstructured staggered grids. Of particular relevance is the way in which finite volume fluxes are related to nodal values. The method is first presented for a simple one-dimensional example of Euler flow in Section 3. Thereafter, more complex cases are discussed in Section 4. The order of accuracy and the size and shape of the discretization can still be chosen freely. An example of a shallow water simulation on a triangulated grid is presented in Section 5. While the current paper was under review, in [23] a similar method (also inspired by [38]) for the shallow water equations was presented.

A large variety of terms may be included in flow equations. For the sake of presentability, this paper discusses only the compressible Euler equations with a conservative force field and a general state equation. This choice of the terms included is motivated by the intended application to the shallow water equations. The shallow water equations are mathematically equivalent to the compressible Euler equations with a specific choice of the state equation. The conservative force field is necessary if water flow over a non-flat bottom profile is to be described.

The paper describes the construction of discretizations inside the computational domain; the discretization of boundaries is outside its scope. The examples presented avoid the problem of boundary condition discretization by applying periodic boundary conditions. Careful analysis of the desired properties of boundary conditions is necessary to obtain discretizations for calculations on domains with actual boundaries; see e.g. [4] for an immersed boundary (cut cell) treatment with application in [13,34].

Complementary to the above structure-preserving attention paid to the space discretization, also the time integration can be reconsidered. In the recent literature several time integration ideas have been presented to preserve discrete energy and/or other invariants, e.g. [28,11,17]. This aspect has been left outside the scope of the present paper. In order for the time integration errors not to 'pollute' the discretization study, time steps have been taken exaggeratingly small.

Remark. The above approach of preserving certain selected quantities during discretization, can also be applied during the process of modeling. E.g. in [35,30] turbulence models are presented that preserve energy (in the absence of diffusion), enstrophy and helicity.

2. Flow equations

2.1. Euler equations

In this section, we present the Euler equations for compressible or incompressible flow, with a conservative force field whose potential is ϕ, given by