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Absorbing boundary conditions for the Euler and Navier–Stokes equations with the spectral difference method

Ying Zhou *, Z.J. Wang

Department of Aerospace Engineering, Iowa State University, 2271 Howe Hall, Ames, IA 50010, United States

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ABSTRACT

Two absorbing boundary conditions, the absorbing sponge zone and the perfectly matched layer, are developed and implemented for the spectral difference method discretizing the Euler and Navier–Stokes equations on unstructured grids. The performance of both boundary conditions is evaluated and compared with the characteristic boundary condition for a variety of benchmark problems including vortex and acoustic wave propagations. The applications of the perfectly matched layer technique in the numerical simulations of unsteady problems with complex geometries are also presented to demonstrate its capability.

1. Introduction

In the last two decades, there have been intensive research efforts on high-order methods for unstructured grids. Such methods provide unprecedented geometric flexibility and accuracy for real world applications. An incomplete list of notable examples includes the spectral element method [51], multi-domain spectral method [43,44], *k*-exact finite volume method [6], WENO methods [27], discontinuous Galerkin (DG) method [7,12,13], high-order residual distribution methods [1], spectral volume (SV) [49,56,64–66] and spectral difference (SD) methods [40,47,50,57,58,67,68]. Spectral difference method originated in the staggered grid multi-domain spectral method [43,44]. Thereafter it was generalized to simplex elements by Liu et al. [47,48]. More recently, a weak instability was discovered by Van Den Abeele et al. [63] and Huynh [41]. The use of Gauss quadrature points and the two ending points as the flux points fixes the problem, and it was proved to be stable by Jameson [42]. A high-order SD method for three-dimensional Navier–Stokes equations on unstructured hexahedral grids developed by Sun et al. [57,58] is used in this paper.

For the numerical simulations of fluid dynamic and aeroacoustic problems, a proper artificial computational boundary condition is needed to minimize the reflection of out-going waves, which can contaminate the physical flow field. This boundary condition is usually called the non-reflecting boundary condition or absorbing boundary condition. It remains a critical component and a difficult challenge in the development of computational fluid dynamics (CFD) and computational aeroacoustics (CAA) algorithms. Significant progresses have been made in this research as reviewed extensively by Colonius [16] and Hu [34].

* Corresponding author. Tel.: +1 515 295 5539.
E-mail address: ying@iastate.edu (Y. Zhou).

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The non-reflecting boundary condition based on the characteristics of the Euler equations was developed as one of the first attempts to minimize the reflection of out-going waves, e.g., in [21,54,61]. In the Godunov-type finite volume methods, the characteristic boundary condition (CBC) based on the linearized one-dimensional Euler equations [45,69,26] is widespread and works well in the numerical simulations of steady problems. For multi-dimensional problems, the performance of the CBC degrades if the wave propagation direction is not aligned with the boundary face normal direction. More efficient and accurate non-reflecting boundary conditions are needed to handle problems like vortex dominated flows and wave propagation problems.

The absorbing boundary condition (ABC) also receives much attention from the electromagnetic and acoustic communities. Engquist and Majda [18] made a pioneering contribution to in this area. Their boundary conditions were constructed to minimize reflections of waves traveling in directions close to perpendicular to the boundary. Higdon [25] further developed the boundary conditions in a simpler and more general form. Another well-known ABC firstly proposed by Bayliss and Turkel [8,9] was developed in an asymptotic expansion of the solution in the far field and annihilate of the leading terms. This type of ABC is widely used in scattering problems to absorb the outgoing disturbances. In the present study, we consider flow problems with strong nonlinear viscous wakes. As a result, we did not pursue the above boundary conditions.

An innovative class of approaches of non-reflecting/absorbing boundary conditions uses extra artificial zones to reduce wave reflections. They are the loosely termed "zonal techniques" [34]. In this type of technique, additional zones surrounding the physical domain are introduced so that in the added zones either the outgoing disturbances/waves are attenuated and thus the reflections are minimized, or the mean flow is altered gradually to be supersonic and thus all disturbances/waves are out-going. Two popular zonal techniques are the absorbing sponge zone (ASZ) technique [19] and perfectly matched layer (PML) technique [39].

The ASZ technique used in this paper was first proposed for the direct acoustic simulation by Colonius et al. [15] and later in a different approach by Ta'asan et al. [59]. It was further developed and theoretically analyzed by Freund [19] and Bodony [11]. Inside the ASZ domain, one source term $-\sigma(Q-\overline{Q})$ is added to the right-hand-side of the governing equations such that the solution Q is gradually tuned to the proposed solution $\overline{Q}$. In order to diminish the reflected error generated at the physical/ASZ interface, the absorbing coefficient σ increases smoothly from zero at the interface to a positive value at the end of the ASZ domain. This technique is widely used in the CAA community for its simplicity and effective performance.

The PML technique was originally developed as an absorbing boundary condition for computational electromagnetic [10] in which the Maxwell equations are numerically solved, and quickly became the method of choice in the computational electromagnetic community [20,53]. The PML equations are formulated such that the amplitude of the out-going waves entering the PML domain can be exponentially reduced while causing as little numerical reflection as possible. It was also found to be a good choice for computational aeroacoustics and computational fluid dynamics [14,22–24,62]. The PML technique was firstly extended to the linearized Euler equations in [28]. However the direct adaption of the split formulation to the Euler equations was found to be unstable in the PML domain [28,29,2–5,60,70]. Hu [30–33,35,52] found that for the PML technique to yield stable absorbing boundary condition, the phase and group velocities of the physical wave supported by the governing equations must be consistent and in the same direction, and a stable formulation of a PML for the linearized Euler equations was proposed. It was proved in [36] that theoretically no reflection will be generated in the PML domain for linearized Euler equations. Further extension of the PML technique to nonlinear Euler and Navier–Stokes equations was given in [37,39]. For nonlinear equations, though the conversion of the equations is not perfectly matched to the original equations due to the nonlinearity of flux vectors, the results of the numerical examples show that the proposed absorbing equations are still effective [37–39].

In this paper, both the ASZ and the PML techniques are extended and implemented for the SD method on hexahedral meshes. The performance and effectiveness are compared with the CBC based on one-dimensional Euler equations. Then a two cylinder case is employed to test the performance of three boundary conditions with complex geometries and vortex dominant flow.

The rest of the paper is organized as follows. In the next section, the formulation of the spectral difference method is briefly reviewed. In Sections 3 and 4, the ASZ and PML approaches used in the spectral difference method are presented with the unstructured hexahedral mesh. In Section 5, numerical tests are presented and discussed. Concluding remarks are given in Section 6.

2. Review of the multi-domain spectral difference method

2.1. Governing equations

Consider the three-dimensional compressible non-linear Navier–Stokes equations written in the conservation form as

$$\frac{\partial Q}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}=0 \tag{2.1a}$$

on domain $\Omega\times[0,T]$ and $\Omega\subset R^3$ with the initial condition

$$Q(x,y,z,0)=Q_0(x,y,z) \tag{2.1b}$$