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Weak solutions and convergent numerical schemes of modified compressible Navier–Stokes equations

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ABSTRACT

Lately, there has been some interest in modifications of the compressible Navier–Stokes equations to include diffusion of mass. In this paper, we investigate possible ways to add mass diffusion to the 1-D Navier–Stokes equations without violating the basic entropy inequality. As a result, we recover Brenner's modification of the Navier–Stokes equations as a special case. We consider Brenner's system along with another modification where the viscous terms collapse to a Laplacian diffusion. For each of the two modifications, we derive a priori estimates for the PDE, sufficiently strong to admit a weak solution; we propose a numerical scheme and demonstrate that it satisfies the same a priori estimates. For both modifications, we then demonstrate that the numerical schemes generate solutions that converge to a weak solution (up to a subsequence) as the grid is refined.

1. Conservation laws

Consider the system of conservation laws in one space dimension:

$$\begin{aligned} &u_t + f(u)_x = 0, \quad x \in \Omega, \quad 0 \le t \le \mathcal{T} \\ &\quad u(x,0) = u^0(x). \end{aligned} \tag{1}$$

Here $u = (u_1, \dots, u_n)^\top$ is the vector of unknowns and the fluxes $f = (f_1, f_2, \dots, f_n)^\top$ are Lipschitz continuous functions of u. Ω is a bounded domain in one dimension (1-D). (We take $\Omega = (0, 1)$.) The system is also subject to appropriate boundary conditions. $\mathcal{T}$ is an arbitrary finite time. $u^0(x)$ is a suitably bounded initial datum.

Conservation laws are often endowed with entropies. Entropy is a useful tool to obtain a priori bounds on the solution and sometimes infer uniqueness. We will briefly introduce the concept. Let (U, F) denote an entropy and entropy flux (for short, entropy pair). By definition $U_u^T f_u = F_u$, and $U_u = w^T$ is termed the entropy variables. Using the entropy variables, (1) can be rewritten as

$$u_w w_t + g(w)_x = 0, \quad x \in \Omega \tag{2}$$

where u_w is symmetric and positive definite and g_w is symmetric. (See [17].)

Often the conservation law is considered to be a model of an associated viscous equation,

$$\begin{aligned} &u_t + f(u)_x = (G(u)u_x)_x, \quad x \in \Omega \\ &\quad u(x,0) = u^0(x), \end{aligned} \tag{3}$$

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where $G(u)$ is a matrix. The regularization $(G(u)u_x)$ is conservative and we will refer to (3) as being conservative. Using the entropy variables, (3) can be stated as

$$u_w w_t + g(w)_x = (\tilde{G}(w)w_x)_x, \quad x \in \Omega. \tag{4}$$

We require that $\tilde{G}$ is symmetric and positive semi-definite. (This property ensures that entropy is diffused.) Note that $\tilde{G}w_x = Gu_x = F^V$ where F^V is commonly known as the viscous flux.

Often, solutions of conservation laws are interpreted in a weak (or averaged) sense. The weak form of (3) is obtained by multiplying the equation by a test function and integrating by parts.

Definition 1.1. A locally integrable function u is defined as a *weak solution* of (3), if it satisfies the following integral identity for all compactly supported test functions $\varphi \in C^\infty(\Omega \times [0, \mathcal{T}))$

$$\int_0^{\mathcal{T}}\int_\Omega (\varphi_t u + \varphi_x f(u) - \varphi_x(Gu_x))\,dx\,dt + \int_\Omega \varphi(x,0)u^0(x)\,dx = 0. \tag{5}$$

Remark. In the case of Ω being periodic, we employ periodic test functions in space.

1.1. The compressible Navier–Stokes equations

In this work, we focus on the special case of gas dynamics. An inviscid gas is governed by the Euler equations, which is a set of conservation laws (1) and in 1-D they take the form,

$$\begin{aligned} u_t + f(u)_x &= 0 \quad x \in \Omega, \quad 0 \le t \le \mathcal{T} \\ u &= (\rho, m, E)^T \\ f(u) &= (m, \rho q^2 + p, (E+p)q)^T \\ p &= (\gamma - 1)(E - \frac{1}{2}\rho q^2). \end{aligned}$$

ρ, q, p and E are the density, velocity, pressure and total energy of a gas. The momentum is denoted as $m = \rho q$ and γ is the ratio of the specific heats. The system is closed using the gas law $p = \rho R T$, where R is the gas constant and T the temperature. In the analysis, we will need the thermodynamic relations, $\gamma = c_p/c_v$ and $R = c_p - c_v$, where c_p and c_v are the specific heat capacities at constant pressure and volume, respectively.

The standard Navier–Stokes equations take the form (3) and are obtained by adding a diffusive flux to the Euler equations.

$$\begin{aligned} u_t + f(u)_x &= (f^{NS})_x \\ f^{NS} &= (0, \frac{4}{3}\mu q_x, \frac{4}{3}\mu q q_x + kT_x)^T \end{aligned} \tag{6}$$

where $\mu > 0$ is the first diffusion coefficient. (We have made the standard assumption that the second diffusion coefficient $\lambda = -2\mu/3$.) $k > 0$ is the thermal diffusivity. These equations are referred to as the Navier–Stokes(–Fourier) (NSF) equations which is the standard set of equations used in compressible viscous fluid dynamics.

In this study, we assume that the domain $\Omega = (0, 1)$, i.e., it is bounded. The system (6) is subject to suitably bounded initial condition $u(x, 0) = u^0(x)$ and we require that $p(x, 0) > 0$ and $\rho(x, 0) > 0$.

Furthermore, the system (6) must be augmented by appropriate boundary conditions. This is a topic in its own right and for simplicity we only consider thermally insulated wall boundary conditions.

$$q = 0|_{\partial\Omega}, \quad T_x = 0|_{\partial\Omega}. \tag{7}$$

Remark. We will use boundary conditions when deriving a priori bounds. However, when considering numerical approximations, we will limit the analysis to the periodic case for simplicity. Demonstrating that it is possible to obtain bounds for the PDE with boundary conditions makes a good case for doing the same with the numerical scheme in the future.

1.2. Background

The standard compressible Navier–Stokes model has been studied extensively and still no general well-posedness results have been obtained. The literature on this subject is vast and we mention only a few results here. In [14], the existence of weak solutions of the *incompressible* equations was proved and in [15] for *isentropic compressible* fluids.