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Optimal energy conserving local discontinuous Galerkin methods for second-order wave equation in heterogeneous media

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ABSTRACT

Solving wave propagation problems within heterogeneous media has been of great interest and has a wide range of applications in physics and engineering. The design of numerical methods for such general wave propagation problems is challenging because the energy conserving property has to be incorporated in the numerical algorithms in order to minimize the phase or shape errors after long time integration. In this paper, we focus on multi-dimensional wave problems and consider linear second-order wave equation in heterogeneous media. We develop and analyze an LDG method, in which numerical fluxes are carefully designed to maintain the energy conserving property and accuracy. Compatible high order energy conserving time integrators are also proposed. The optimal error estimates and the energy conserving property are proved for the semi-discrete methods. Our numerical experiments demonstrate optimal rates of convergence, and show that the errors of the numerical solutions do not grow significantly in time due to the energy conserving property.

1. Introduction

Wave propagation is a fundamental form of energy transmission, which arises in many fields of science, engineering and industry, and it is significant to geoscience, petroleum engineering, telecommunication, and the defense industry (see [23,32] and the references therein). Efficient and accurate numerical methods to solve wave propagation problems are of fundamental importance to these applications. Experience reveals that energy conserving numerical methods, which conserve the discrete approximation of the energy, are favorable because they are able to maintain the phase and shape of the waves accurately, especially for long time simulation. In [43], we have designed a high order accurate energy conserving local discontinuous Galerkin (LDG) method for the one-dimensional second-order wave equation with constant coefficient.

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In this paper, we focus on multi-dimensional problems in heterogeneous media, and develop optimally convergent LDG methods which also conserve the energy in the discrete sense.

The wave equation can be written in a second-order form, or an equivalent first-order hyperbolic system. Directly solving the second-order equation usually involves fewer unknown variables, therefore the resulting numerical schemes are more efficient. This saving can be significant in the three-dimensional applications. For example, for the linear elasticity equations in three dimensions, three variables are used in the second-order form, while first-order system needs at least nine components [6]. In addition, there are many applications where the second-order PDEs arise naturally. When converted into first-order systems, they may admit a wider class of solutions, therefore some constraints are needed to ensure that these solutions are solutions of the original second-order equation, which also increases difficulty to the design of numerical methods. Finally, it was also shown [3] that the second-order equation may allow larger time step size, compared to the first-order system.

A vast amount of literature can be found on the numerical approximations of the second-order wave equation. The most common numerical method for solving the wave equation is to use the second order accurate centered finite difference operator. One major component in designing such finite difference methods which conserve the energy numerically is the Summation By Parts (SBP) operator, with special attention paid near the boundaries. There have been many studies on this subject (see [42] and the references therein). While finite difference methods provide efficient solvers, they are largely limited by the geometry of the domain, although some attempts [4] have been made to circumvent this difficulty. In contrast, finite element methods have the flexibility in handling complex geometry. Safjan and Oden [40] introduced a family of unconditionally stable high order Taylor–Galerkin schemes for acoustic and elastic wave propagation. Faccioli et al. [24] used explicit Fourier–Legendre domain decomposition methods and focused on the numerical validation of the methods. Spectral methods for acoustic and elastic waves have been developed in [35,45], and a mortar coupling between spectral and finite elements methods for elastodynamic problem on complex geometries can be found in [9]. Spectral element methods are shown to conserve energy when applied to the wave equations [2,27]. We refer to [19,28] for a review of previous work on spectral and spectral element methods. Here, we will confine our attention in discontinuous Galerkin (DG) methods, which have the advantages of being local (versus global), easy h-p adaptivity and being able to handle hanging nodes, compared with spectral element methods. DG methods can be viewed as spectral element methods with domain decomposition. They belong to a class of finite element methods using discontinuous piecewise polynomial spaces for both the numerical solution and the test functions. They were originally devised to solve hyperbolic conservation laws with only first order spatial derivatives, e.g. [13–15,17,18]. They allow arbitrarily unstructured meshes, and have compact stencils. Moreover, they easily accommodate arbitrary h-p adaptivity. DG methods were later generalized to the LDG methods by Cockburn and Shu to solve convection–diffusion equations [16], motivated by successful numerical experiments from Bassi and Rebay [7] for the compressible Navier–Stokes equations. Recently, Zhong and Shu [46] studied the question of how many grid points (degrees of freedom) per wave length are needed to achieve a given accuracy for the DG method applied to the linear wave equation, following the classical error analysis by Kreiss and Oliger [34] for the finite difference methods.

Many DG methods have been developed for the wave equation in both first-order and second-order forms [1,5,15,25,30, 37,39,40], and some of these methods are also energy conserving [11,26,29]. Two approaches are commonly used to achieve the energy conserving property. The first one is to introduce two staggered mesh sets, and define one set of solution on each mesh. This usually leads to more complexity, as staggered mesh may be difficult to construct, especially for high dimensional complex domain and in the neighborhood of the boundary. Recently, Chung and Engquist [11,12] have proposed an optimal, energy conserving DG method for the first-order wave equation using staggered grids. They introduced different meshes for different computational variables, and are able to prove the optimal convergence for unstructured meshes. The other approach to obtain energy conserving method is to use the central numerical flux [26], i.e., the numerical flux along cell boundaries is evaluated by taking the average of two values of the numerical solution from the two neighboring cells. However, only suboptimal convergence can be proven theoretically, and numerically, one can observe optimal convergence if even order polynomial space is used, and suboptimal if odd order polynomial space is used.

Usually it is difficult to obtain DG schemes for wave equations which are non-dissipative (energy conserving for the physical energy) and optimal high order accurate. In [43], we have designed an energy conserving LDG method for the simple one-dimensional second-order constant coefficient wave equation. We have proved that the proposed method has the optimal convergence rates in both the energy and L^2 norms, and the upper bound of the errors grows in time only in a linear fashion. In this paper, we consider the multi-dimensional wave problems in heterogeneous media. Extension of the previous work to the multi-dimensional problems on Cartesian meshes is discussed. Extra attention needs to be paid at the interface of different media to ensure the stability and energy conservation. Theoretical proof, as well as the numerical evidence, indicates that a good choice of the projection of the initial condition into the polynomial space is important to achieve optimal convergence rate. The semi-discrete LDG method will be coupled with high order explicit energy conserving time discretization. We remark here that since our scheme is non-dissipative, it is more oscillatory than the commonly used upwind (energy-dissipative) DG method when applied to problems with discontinuities. The advantage of energy conserving methods is to solve smooth wave problems, with the attempt to resolve all waves for long time periods.

The outline of our paper is as follows. In Section 2, we present the semi-discrete LDG method, and prove its energy conserving property. The optimal error estimates, both in the energy norm and the L^2 norm, are analyzed in Section 3, and therein, the upper bound of errors is proved to grow linearly in time. The fully discrete LDG method, with the high order energy conserving time discretization, and its energy conserving properties are presented in Section 4. Section 5