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Second order symmetry-preserving conservative Lagrangian scheme for compressible Euler equations in two-dimensional cylindrical coordinates

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ABSTRACT

In applications such as astrophysics and inertial confinement fusion, there are many three-dimensional cylindrical-symmetric multi-material problems which are usually simulated by Lagrangian schemes in the two-dimensional cylindrical coordinates. For this type of simulation, a critical issue for the schemes is to keep spherical symmetry in the cylindrical coordinate system if the original physical problem has this symmetry. In the past decades, several Lagrangian schemes with such symmetry property have been developed, but all of them are only first order accurate. In this paper, we develop a second order cell-centered Lagrangian scheme for solving compressible Euler equations in cylindrical coordinates, based on the control volume discretizations, which is designed to have uniformly second order accuracy and capability to preserve one-dimensional spherical symmetry in a two-dimensional cylindrical geometry when computed on an equal-angle-zoned initial grid. The scheme maintains several good properties such as conservation for mass, momentum and total energy, and the geometric conservation law. Several two-dimensional numerical examples in cylindrical coordinates are presented to demonstrate the good performance of the scheme in terms of accuracy, symmetry, non-oscillation and robustness. The advantage of higher order accuracy is demonstrated in these examples.

1. Introduction

The class of Lagrangian methods, which have the mesh moving with the local fluid velocity, is one of the main classes of numerical methods for simulating multi-material fluid flows. Since it has the distinguished advantage in capturing material interfaces automatically and sharply, it is widely used in many fields for multi-material flow simulations such as astrophysics, inertial confinement fusion (ICF) and computational fluid dynamics (CFD).

In many application fields such as ICF and astrophysics, there exist many three-dimensional cylindrical-symmetric models such as sphere-shape capsules and cylinder-shape hohlraum. This kind of models is usually simulated by Lagrangian methods in the two-dimensional cylindrical coordinates. For a Lagrangian scheme applied in these problems, one critical issue

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is how to maintain the spherical symmetry property in a cylindrical coordinate system, if the original physical problem has this symmetry. For example, in the simulation of implosions, a small deviation from spherical symmetry caused by numerical errors may be amplified by physical or numerical instabilities which may lead to unpredicted large errors. Numerous studies exist in the literature on this issue. In the past several decades, first order spherical-symmetry-preserving Lagrangian methods in two-dimensional cylindrical coordinates were well investigated. Among them, the most widely used method that keeps spherical symmetry exactly on an equal-angle-zoned grid in cylindrical coordinates is the area-weighted method, e.g. [20,25,24,2,4,15,1,16,19]. In this approach, in order to obtain the spherical symmetry property, a Cartesian form of the momentum equation is used in the cylindrical coordinate system, hence integration is performed on area rather than on the true volume in cylindrical coordinates. The main flaw of this kind of area-weighted schemes is that they may not maintain conservation for the momentum. Differently from the area-weighted schemes, Browne [3] presented a Lagrangian scheme termed "integrated total average" which is discretized on the true control volume. The scheme has been proven to be able to preserve the desired spherical symmetry in the two dimensional cylindrical geometry for equal-angle zoning. Unfortunately, this scheme cannot keep the conservation of momentum and total energy either. Margolin and Shashkov used a curvilinear grid to construct symmetry-preserving discretizations for Lagrangian gas dynamics [17]. In our recent work [7], a cell-centered Lagrangian scheme has been developed which is based on the control volume discretization. By compatible discretizations of the source term in the momentum equation, the scheme is designed to preserve one-dimensional spherical symmetry in a two-dimensional cylindrical geometry when computed on an equal-angle-zoned initial grid. A distinguished feature of the scheme is that it can keep both the properties of symmetry and conservation. In [8], we apply the methodology proposed in [7] to the first order control volume scheme of Maire in [15] to obtain the spherical symmetry property. The modified scheme can preserve several good properties such as symmetry, conservation and the geometric conservation law (GCL).

Although the issue on the symmetry-preserving property of the Lagrangian schemes has been well investigated, the situation is less satisfactory in terms of accuracy. Up to now, all the existing symmetry-preserving Lagrangian schemes are only first order accurate. In fact, to design a scheme with the preservation of spherical symmetry, not only the nodal velocity but also all the variables appearing in the integral of the numerical flux should be calculated symmetrically, which is especially difficult for a high order scheme to achieve. Moreover, a careful treatment must be performed on the source term, which has been the biggest obstacle for a Lagrangian scheme to be symmetry-preserving, even in the case of first order accuracy, and this is exactly the reason for most people to adopt the area-weighted schemes. It is quite challenging to design a higher than first order Lagrangian scheme with both the properties of spherical symmetry and conservation.

In this paper, we design a second order cell-centered Lagrangian scheme for solving Euler equations of compressible gas dynamics in cylindrical coordinates. The scheme is based on the control volume discretizations. It is uniformly second order accurate and is able to preserve one-dimensional spherical symmetry in a two-dimensional cylindrical geometry when computed on an equal-angle-zoned initial grid, and meanwhile it has many other good properties such as conservation for mass, momentum and total energy and the GCL. Several two-dimensional numerical examples in cylindrical coordinates are presented to demonstrate the good performance of the scheme in terms of accuracy, symmetry, non-oscillation and robustness. The advantage of higher order accuracy is demonstrated in the numerical examples.

An outline of the rest of this paper is as follows. In Section 2, we describe our new second order symmetry-preserving Lagrangian scheme in two-dimensional cylindrical coordinates. In Section 3, we prove the symmetry-preserving property of the scheme. In Section 4, numerical examples are given to demonstrate the performance of the new Lagrangian scheme. In Section 5 we will give concluding remarks.

2. The design of a second order cell-centered symmetry-preserving Lagrangian scheme in cylindrical coordinates

2.1. The compressible Euler equations in a Lagrangian formulation in cylindrical coordinates

The compressible inviscid flow is governed by the Euler equations in the cylindrical coordinates which have the following integral form in the Lagrangian formulation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\iint_{\Omega(t)} \rho r \mathrm{d}r \mathrm{d}z = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\iint_{\Omega(t)} \rho u_z r \mathrm{d}r \mathrm{d}z = -\int_{\Gamma(t)} P n_z r \mathrm{d}l, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\iint_{\Omega(t)} \rho u_r r \mathrm{d}r \mathrm{d}z = -\int_{\Gamma(t)} P n_r r \mathrm{d}l + \iint_{\Omega(t)} P \mathrm{d}r \mathrm{d}z, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\iint_{\Omega(t)} \rho E r \mathrm{d}r \mathrm{d}z = -\int_{\Gamma(t)} P \mathbf{u}\cdot\mathbf{n} r \mathrm{d}l, \end{cases} \tag{2.1}$$

where z and r are the axial and radial directions respectively. ρ is density, P is pressure, E is the specific total energy. $\mathbf{u} = (u_z, u_r)$ where u_z and u_r are the velocity components in the z and r directions respectively, and $\mathbf{n} = (n_z, n_r)$ is the unit