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A high-order mimetic method on unstructured polyhedral meshes for the diffusion equation

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ABSTRACT

We present a new family of mimetic finite difference schemes for solving elliptic partial differential equations in the primal form on unstructured polyhedral meshes. These mimetic discretizations are built to satisfy local consistency and stability conditions. The consistency condition is an exactness property, i.e., the mimetic schemes are exact when the solution is a polynomial of an assigned degree. The stability condition ensures the well-posedness of the method. The degrees of freedom are the solution moments on mesh faces and inside mesh cells. Higher order schemes are built using higher order moments. The developed schemes are verified numerically on diffusion problems with constant and spatially variable (possibly, discontinuous) tensorial coefficients.

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1. Introduction

The mimetic discretization framework has been developed to solve PDEs on arbitrary polygonal and polyhedral meshes [11]. In contrast to finite volume methods [22] that can also handle general meshes, it has a solid mathematical foundation based on a discrete vector and tensor calculus [28]. Thus, the resulting discrete schemes preserve or mimic important properties of continuum PDEs such as symmetry and positivity of discrete operators, exact discrete identities, and discrete Helmholtz space decompositions. In this paper, we exploit flexibility of the mimetic framework to mix and match degrees of freedom of various nature to develop a new family of mimetic finite difference (MFD) schemes for the diffusion equation.

The development of the MFD method has a long history. It has been applied successfully to a wide range of scientific and engineering problems, such as continuum mechanics [33], discretization of differential forms [14,34], electromagnetics [16,25,27], gas dynamics [18], linear diffusion equation [2,12,17,20,26,29,30], convection–diffusion equation [6,21], steady Stokes equations [7–10], elasticity [3], elliptic obstacle [1], Reissner–Mindlin plates [13], eigenvalues [19] and two-phase flows in porous media [31].

Due to the importance of discrete conservations laws for engineering simulations, the MFD schemes were originally developed for systems of PDEs formulated using first-order differential operators. The discrete operators are constructed in pairs: first principles are used to discretize one operator (e.g. primary gradient, divergence, and curl), and discrete duality relationships are used to derive the second discrete operator (respectively, derived divergence, gradient, and curl).

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In [15], it was realized that two underlying discretization principles guiding construction of the derived mimetic operators, named consistency and stability, can be applied to primal formulations of PDEs. More precisely, the resulting MFD method builds a discrete bilinear form $\mathcal{A}_h$ that approximates the continuum form $\mathcal{A}$ and preserves its kernel. This construction is done cell-by-cell like in the finite element method (FEM). In contrast to the FEM, there is no analog of a unisolvency condition that often makes construction of high-order elements a non-trivial task. The mimetic framework allows us to use variable number of degrees of freedom per cell that can be associated with different geometric objects and have different meaning (e.g. pointwise values, moments, or projection of vector functions on face normals). Usage of additional degrees of freedom to simplify the construction violates the unisolvency condition but does not break the convergence and stability properties. It simply leads to a parametric family of mimetic schemes that can be further analyzed for existence of schemes with additional properties such as the discrete maximum principle [29].

In [9], the consistency and stability conditions are developed for arbitrary-order mimetic schemes for elliptic equations in two dimensions. The consistency condition is formulated as an exactness property for polynomials of a given degree. On its turn, the stability condition enforces the coercivity of the discrete bilinear form and, eventually, the well-posedness of the resulting mimetic scheme. Extension of these schemes to three dimensions requires the construction of high-order quadrature rules for polygonal faces of polyhedral cells. Such quadrature rules are not available for an arbitrary polygon and their numerical construction will make the method too expensive. The polygonal FEMs also suffers of this issue [32,35–39], which is overcome only by the Galerkin reformulation of the mimetic methods, recently proposed as "*the virtual element method*" in [4].

In this paper, we resolve this issue using a special choice of the degrees of freedom. Instead of using nodal degrees of freedom, which may be associated with either the mesh vertices and other special nodes on the cell interfaces, we use solution moments on faces and inside cells. The construction requires to calculate moments of only polynomial functions which is a problem with a well-known solution. The new mimetic schemes are suitable to the numerical approximation of two- and three-dimensional elliptic problems at any order of accuracy on an arbitrary polygonal or polyhedral mesh. It is worth mentioning that this kind of approximation shares many characteristics with the non-conforming finite element method, and, for this reason, we might refer to it as *the non-conforming mimetic method*.

The paper is organized as follows. In Section 2, we introduce the model problem, we briefly review the basic concepts of mimetic discretizations, and we describe the construction of the new non-conforming mimetic method. In Section 3 we prove the consistency and stability of the proposed mimetic method. In Section 4 we discuss two special cases where the general formulation is significantly simpler: the low-order approximation and the non-conforming mimetic method (of any order of accuracy) for problems with constant diffusion tensors. In Section 5, we study the performance of the proposed method for the numerical resolution of problems with smooth and discontinuous coefficients. Final conclusions are in Section 6.

2. The mimetic finite difference method

In this section, we introduce the model problem, we give a brief overview of the basic concepts of the MFD method, and we describe the construction of the new non-conforming mimetic method.

2.1. Model problem

The steady diffusion problem for the scalar solution field u is governed by the following equation:

$$-\operatorname{div}(\mathsf{K}\nabla u) = f \quad \text{in } \Omega, \tag{1}$$

$$u = g \quad \text{on } \Gamma, \tag{2}$$

where $\Omega \subset \mathbb{R}^3$ is a polyhedral domain with Lipschitz boundary Γ, K is a diffusion tensor describing material properties, $f \in L^2(\Omega)$ is a given loading term, and $g \in H^{\frac{1}{2}}(\Gamma)$ is a given function. We assume that K is a bounded, measurable, and symmetric tensor in $(W^{1,\infty}(\Omega))^{3\times 3}$. We also assume that K is *strongly elliptic*, i.e., there exist two positive constants κ_* and κ^* such that

$$\kappa_* \|\mathbf{v}\|^2 \leqslant \mathbf{v} \cdot \mathsf{K}(\mathbf{x})\mathbf{v} \leqslant \kappa^* \|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbb{R}^3,\ \forall \mathbf{x} \in \Omega,$$

where $\|\mathbf{v}\|$ is the Euclidean norm of vector $\mathbf{v}$. The strong ellipticity implies that matrix $\mathsf{K}(\mathbf{x})$ is strictly positive definite and thus non-singular for every $\mathbf{x} \in \Omega$.

We consider the affine subspace of $H^1(\Omega)$,

$$\mathcal{V}_g = \{ v \in H^1(\Omega) \mid v_{|\Gamma} = g \},$$

and the linear subspace $\mathcal{V}_0$ for $g = 0$. Let us introduce the bilinear form

$$\mathcal{A}(u, v) = \int_\Omega \mathsf{K}\nabla u \cdot \nabla v \, dV \tag{3}$$