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Finite-difference schemes for anisotropic diffusion

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ABSTRACT

In fusion plasmas diffusion tensors are extremely anisotropic due to the high temperature and large magnetic field strength. This causes diffusion, heat conduction, and viscous momentum loss, to effectively be aligned with the magnetic field lines. This alignment leads to different values for the respective diffusive coefficients in the magnetic field direction and in the perpendicular direction, to the extent that heat diffusion coefficients can be up to 10^{12} times larger in the parallel direction than in the perpendicular direction. This anisotropy puts stringent requirements on the numerical methods used to approximate the MHD-equations since any misalignment of the grid may cause the perpendicular diffusion to be polluted by the numerical error in approximating the parallel diffusion. Currently the common approach is to apply magnetic field-aligned coordinates, an approach that automatically takes care of the directionality of the diffusive coefficients. This approach runs into problems at *x*-points and at points where there is magnetic re-connection, since this causes local non-alignment. It is therefore useful to consider numerical schemes that are tolerant to the misalignment of the grid with the magnetic field lines, both to improve existing methods and to help open the possibility of applying regular non-aligned grids. To investigate this, in this paper several discretization schemes are developed and applied to the anisotropic heat diffusion equation on a non-aligned grid.

1. Introduction

Anisotropic diffusion is a common physical phenomenon and describes processes where the diffusion of some scalar quantity is direction dependent. Anisotropic diffusive processes are for instance transport in porous media, large-scale turbulence where turbulence scales are anisotropic in size, and of interest to us: heat conduction and momentum dissipation in fusion plasmas.

In tokamak fusion plasmas the viscosity and heat conduction coefficient parallel to the magnetic field may be in the order of 10^6 and 10^{12} times larger, respectively, than perpendicular conduction coefficients. This is caused by the fact that the heat conductivities parallel and perpendicular to the magnetic field lines are determined by different physical processes; along the field lines particles can travel relatively large distances without collision whereas perpendicular to the field lines the mean free path is in the order of the gyroradius, see e.g. Hölzl [37].

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Numerically, high anisotropy may lead to the situation that errors in the direction of the largest diffusion coefficient may significantly influence the diffusion in the perpendicular direction. This necessitates a high-order approximation in the direction of the largest coefficient value (see e.g. Sovinec et al. [65], Meier et al. [54]). Given the high level of anisotropy in tokamak plasmas, a numerical approximation may introduce large perpendicular errors if the magnetic field direction is strongly misaligned with the grid. Problems that may arise with highly anisotropic diffusion problems on non-aligned meshes are in general:

- significant numerical diffusion perpendicular to the magnetic field lines due to grid misalignment, see e.g. Umansky et al. [66],
- non-positivity near high gradients, see e.g. Sharma and Hammett [63],
- mesh locking, stagnation of convergence dependent on anisotropy, see e.g. Babuška and Suri [8],
- convergence loss in case of variable diffusion tensor, see e.g. Günter et al. [31].

It is possible to use a field-aligned coordinate system. However this cannot be maintained throughout the plasma; problems arise at x-points and in regions of highly fluctuating magnetic field directions (for instance in case of edge turbulence). To confidently perform simulations of phenomena that rely heavily on the resolution of the perpendicular temperature gradient we must apply a scheme that maintains sufficient accuracy in case of varying anisotropy and misalignment.

The bulk of the present methods are designed with discontinuous diffusion tensors in mind, and often on general and distorted non-uniform grids. We give an (inexhaustive) overview of methods used today, for details the reader is referred to the specific papers.

We start with the Multi-Point Flux Approximation (MPFA), a cell-centered finite volume method commonly used for approximating anisotropic diffusion with discontinuous tensors on distorted meshes, see e.g. Aavatsmark et al. [2–5], and Edwards and Rogers [25]. The MPFA uses cell-centered unknowns and connects the volumes using shared subcells with a local low-order interpolation of the primary unknowns. The method is robust in terms of diffusion tensor discontinuity as it is locally conservative, but the resulting diffusion operator is often non-symmetric and formal accuracy cannot be maintained for higher levels of anisotropy. The MPFA method comes in various flavors, depending on how the fluxes are approximated, for instance the original MPFA-O and MPFA-U methods by Aavatsmark et al. [2,3] and more recently by Aavatsmark et al. [1] and Agélas et al. [6] respectively, the symmetric MPFA-L and MPFA-G methods.

Le Potier [43] devised a cell-centered finite-volume method where the gradients are solved on each vertex by imposing flux continuity conditions, similar to the MPFA approach. Eymard et al. [26] devised a cell-centered finite-volume scheme using a special discrete gradient operator. Maire and Breil [11,52] apply an MPFA-like finite-volume method with cell-centered unknowns and a local variational formulation to obtain the fluxes in their Cell-Centered Lagrangian Diffusion (CCLAD) approach, with the requirement that temperature and sub-face normal fluxes are continuous. Maire and Breil [53] also constructed a CCLAD method where the fluxes are constructed using finite differences. Jacq et al. [40] expanded the method to three dimensions.

Le Potier and Ong [45] and Ong [59] devised a cell-centered method which makes use of a dual grid. The dual grid unknowns are chosen to be linear combinations of cell unknowns. This so-called Finite Element Cell-Centered (FECC) method uses less unknowns per cell compared to other dual grid methods which apply both cell-centered unknowns and cell–face or vertex unknowns. Another difference is the use of a third grid which is a sub-grid of the dual grid. The formal convergence of the FECC method seems to be maintained for discontinuous diffusion tensors with large values for the anisotropy [45].

Shashkov and Steinberg [64] constructed the Support Operator Method (SOM), also known as Mimetic Finite Difference (MFD) methods. Hyman et al. [38,39] and Brezzi et al. [12,13] apply and categorize the MFD methods. The MFD methods are mimetic to the extent that they preserve the self-adjointness of the divergence and the flux operator. Key to the MFD methods is the use of a dual grid, where flux values and temperature values are placed on separate grid points, and the application of a variational formulation to find the flux values, such that the self-adjointness between the discrete divergence operator and the discrete gradient operator is guaranteed. Downside of the original MFD schemes is the use of non-local operators. Formal convergence is robust for high levels of anisotropy, grid non-uniformity and discontinous diffusion tensors. Further, the diffusion operator is symmetric positive definite. Günter et al. [31] apply the MFD method to fusion plasma relevant test cases and maintain the order of accuracy for non-aligned (regular, rectangular) meshes. Günter et al. [30] apply the support-operator approach from Hyman et al. [39] to a finite-element method. The method is adapted to have a local flux description by Morel et al. [56], which requires both cell-centered and face-centered unknowns. The MFD method is finally made local and cell-centered by Lipnikov et al. [50] and Lipnikov and Shashkov [48].

Hermeline [33,34] uses a dual grid, solving the diffusion equation on each grid where the temperature and the diffusion tensor values are defined in the same nodes. This is termed the Discrete Duality Finite-Volume (DDFV) method. The DDFV method requires the solution of the diffusion equation on two meshes and as such requires more unknowns. The resulting matrices are positive definite. Formal convergence for highly anisotropic problems (with the ratio between parallel and perpendicular diffusion coefficient 10^{12}) is close to second order for higher resolutions but not anisotropy-independent for coarser grids, see Le Potier and Ong [45]. The FECC method bares resemblance to the DDFV method where the former uses a third subgrid and cell-centered unknowns.

Other methods involving the use of dual grids are the Hybrid Finite Volume method (HFV) and the Mixed Finite Volume (MFV) method, see Eymard et al. [27] and Droniou and Eymard [23] respectively. Droniou et al. [24] formally proved the