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Steady discrete shocks of 5th and 7th-order RBC schemes and shock profiles of their equivalent differential equations

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ABSTRACT

An exact expression of steady discrete shocks was recently obtained by the author in [9] for a class of residual-based compact schemes (*RBC*) applied to the inviscid Bürgers equation in a finite domain. Following the same lines, the analysis is extended to an infinite domain for a scalar conservation law with a general convex flux. For the dissipative high-order schemes considered, discrete shocks in infinite domain or with boundary conditions at short distance (Rankine–Hugoniot relations) are found to be very close. Besides, the present analytical description of shock capturing in infinite domain is explicit and so simple that it could lead to a new approach for correcting parasitic oscillations of high order *RBC* schemes. In a second part of the paper, exact solutions are also derived for equivalent differential equations (*EDE*) approximating RBC_{2p-1} schemes (subscript denotes the accuracy order) at orders $2p$ and $2p+1$. Although *EDE* involves Taylor expansions around steep structures, agreement between the exact *EDE* shock-profiles and the discrete shocks is remarkably good for RBC_5 and RBC_7 schemes. In addition, a strong similarity is demonstrated between the analytical expressions of discrete shocks and *EDE* shock profiles.

1. Introduction

The development of high-order numerical schemes for solving nonlinear hyperbolic problems requires efficient tools to study and improve their shock capturing capabilities. Discrete shocks were first analyzed in the seventies by Jennings [1] for monotone schemes approximating a single conservation law and by Majda and Ralston [2] for dissipative first-order schemes applied to hyperbolic systems in the case of weak shocks. They proved the existence and stability of discrete shock profiles. Later the theory was further developed, notably by Michelson [3], Smyrlis [4], Serre [5], Arora and Roe [6], Bultelle, Grassin and Serre [7] and Benzoni-Gavage [8]. Recently the author [9] proposed an exact expression of steady discrete shocks for a class of residual-based compact (*RBC*) schemes applied to the inviscid Bürgers equation on a finite domain. In an *RBC* scheme, the numerical dissipation as well as the consistent part is expressed only in terms of approximations of the complete residual r containing all the terms in the governing equations including the time derivative. More precisely, the numerical dissipation involves space first-derivatives of r. Approximations of r are made using Pade fractions of discrete operators. On a Cartesian mesh, *RBC* schemes can approximate a hyperbolic system of conservation laws in d-dimension with a 5th or 7th-order accuracy on a 5^d-point stencil. For these odd accuracy-orders, the leading error (of order 5 or 7) is dissipative and dominates the dispersive error (of order 6 or 8), which is a favorable feature for robustness. Description and analysis of the *RBC* schemes can be found in [10–15]. A related approach developed on unstructured meshes concerns

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the residual-distribution schemes of Abgrall, Deconinck and Ricchiuto [16–19] for which the residuals are distributed to the nodes of triangles or tetrahedrons.

The purpose of the present paper is twofold. First it is devoted to the extension of the discrete shock analysis to an infinite domain for a scalar conservation law with a general convex flux. Comparison of exact solutions in infinite domain and in finite domains of various size, even as small as the shock thickness, shows very little difference between the discrete shocks. Besides, the mathematical expression of discrete shocks in infinite domain is so simple that it could be used in the future studies for correcting parasitic oscillations of very high-order *RBC* schemes. The second part of the paper concerns the validity of the equivalent differential equation (also called modified equation or differential approximation) for representing discrete shocks of high order schemes. Given a numerical scheme accurate at order p, its equivalent differential equation at order q ($q > p$) is a differential equation approximated by the scheme at order q. Considered as a model of the scheme, this equation has been successfully used for different purposes such as linear and nonlinear stability analysis by Hirt [20], Yanenko and Shokin [21], Warming and Hyett [22] and Majda and Osher [23] and control of parasitic oscillations by Chin [24], Hedstrom [25] and Lerat and Peyret [26,27], see also the text books [28–30]. However, the relevance of the approach for representing discrete shocks has not been established in a general situation. The most advanced work on this topic is due to Goodman and Majda [31] who proved the validity of the equivalent differential equation for a traveling shock of an upwind scheme approximating a scalar conservation law with the special nonlinear flux: $f(w) = -\log(\beta + \gamma e^{-w})$, $\beta > 0$, $\gamma = 1 - \beta$. In the present study, some more light is shed on this validity by deriving exact solutions to equivalent differential equations of high-order *RBC* schemes for a convex flux. This study also reveals an analogy between the analytical expressions of shock profiles of equivalent differential equations and discrete shocks.

The paper is organized as follows. Section 2 briefly reminds the RBC_5 and RBC_7 schemes for solving a hyperbolic system of conservation laws in one space-dimension. Section 3 describes the construction of the exact solution of the steady discrete shock problem in an infinite domain and compare it to the solution in finite domains of various size. Then Section 4 presents the equivalent differential equations of RBC_{2p-1} schemes at orders $2p$ and $2p+1$ and derives their steady shock profiles. Section 5 compares these shock profiles to the discrete shocks for $p = 3$ and 4 and exhibits the similarity between their mathematical expressions. Conclusions are drawn in Section 6.

2. One-dimensional RBC_5 and RBC_7 schemes

For the one-dimensional hyperbolic system

$$\frac{\partial w}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{1}$$

where t is the time, x the space variable, w the state vector and $f = f(w)$ a smooth flux-function, an *RBC* scheme is a compact discrete form of

$$\frac{\partial w}{\partial t} + \frac{\partial f}{\partial x} = \frac{\delta x}{2}\frac{\partial}{\partial x}\left[\phi\left(\frac{\partial w}{\partial t} + \frac{\partial f}{\partial x}\right)\right] \tag{2}$$

where ϕ is a numerical viscosity matrix depending only of the Jacobian-matrix $A = df/dw$ and δx is a spatial step. More precisely, consider a uniform mesh $x_j = j\delta x$ and introduce the classical difference and average operators:

$$(\delta v)_{j+\frac{1}{2}} = v_{j+1} - v_j \qquad (\mu v)_{j+\frac{1}{2}} = \frac{1}{2}(v_{j+1} + v_j)$$

where v_j is a mesh function, j being integer or half integer.

The spatial approximation of (2) is

$$\tilde{r}_j = \frac{1}{2}\left[\delta(\phi\tilde{r}_1)\right]_j \tag{3}$$

where $\tilde{r}_j$ and $(\tilde{r}_1)_{j+\frac{1}{2}}$ are centered compact approximations of the exact residual $r = \partial w/\partial t + \partial f/\partial x$.

Limiting the scheme stencil to 5 points, these approximations can be written as:

$$\tilde{r}_j = \left[\left(I + \bar{b}\,\delta^2 + \bar{c}\,\delta^4\right)\frac{\partial w}{\partial t} + \left(I + \bar{a}\,\delta^2\right)\frac{\delta\mu f}{\delta x}\right]_j \tag{4}$$

$$(\tilde{r}_1)_{j+\frac{1}{2}} = \left[\left(I + a^{\mu}\,\delta^2\right)\mu\frac{\partial w}{\partial t} + \left(I + a^{\delta}\,\delta^2\right)\frac{\delta f}{\delta x}\right]_{j+\frac{1}{2}} \tag{5}$$

Using the coefficients given in Table 1, we obtain dissipative schemes spatially accurate at order 5 and 7, denoted as RBC_5 and RBC_7 (see [12,13]).

The time approximation can also be made at various accuracy orders but does not matter in this paper devoted to the analysis of steady-state solutions. When such a solution is reached, the scheme (3) with (4) and (5) reduces to