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Short note

Convecting reference frames and invariant numerical models

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ABSTRACT

In the recent paper by Bernardini et al. [1] the discrepancy in the performance of finite difference and spectral models for simulations of flows with a preferential direction of propagation was studied. In a simplified investigation carried out using the viscous Burgers equation the authors attributed the poorer numerical results of finite difference models to a violation of Galilean invariance in the discretization and propose to carry out the computations in a reference frame moving with the bulk velocity of the flow. Here we further discuss this problem and relate it to known results on invariant discretization schemes. Non-invariant and invariant finite difference discretizations of Burgers equation are proposed and compared with the discretization using the remedy proposed by Bernardini et al.

1. Introduction

In the recent paper [1] a possible remedy was discussed to improve the poor numerical behavior of finite difference simulations of turbulent flows with a preferential propagation direction. It was shown that the violation of Galilean invariance of the finite difference scheme is the most likely explanation why it is necessary to use a significantly larger number of grid points in finite difference calculations than in spectral methods to achieve comparably accurate numerical results. The recommendation given in [1] is to carry out the finite difference computations in a reference frame that moves with the constant stream-wise bulk velocity in the flow direction. It was then shown for the example of Burgers equation that the finite difference model may yield similar numerical results as spectral discretizations with approximately the same number of grid points.

In the present paper we further discuss this problem and the remedy proposed in [1]. In fact, the problem found and analyzed in [1] has been investigated quite intensively in the field of group analysis of differential and difference equations, see e.g. [2,3,10,11,14,15,19] and references therein for some of the most recent results. In particular, it was established by Dorodnitsyn and collaborators [6,11,12] that it is not possible to maintain the Galilean invariance of partial differential equations in a finite difference model when the mesh does not move in the course of the numerical integration. This result qualitatively explains why the method proposed in [1] may work from the geometrical point of view.

The violation of Galilean invariance of stationary discretizations can be readily shown by applying a Galilean boost, which in the one-dimensional case is

$$(\tilde{t}, \tilde{x}, \tilde{u}) = (t, x + \varepsilon t, u + \varepsilon), \tag{1}$$

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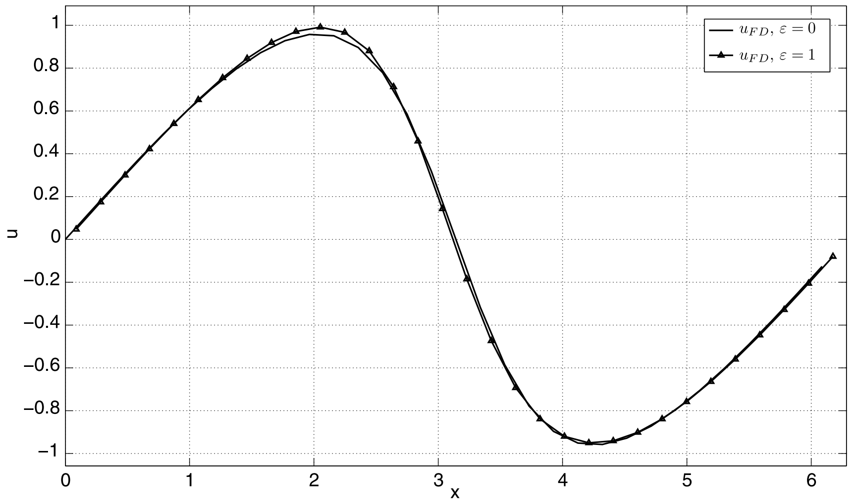

Fig. 1. Integration using the classical FTCS discretization of Burgers equation (2). *Solid line:* Original integration in a resting reference frame. *Solid lines with triangles:* Integration in a reference frame moving with constant velocity $\varepsilon = 1$ as in [1]. The results in the moving reference frames were shifted back to the origin for proper comparison.

where $\varepsilon \in \mathbb{R}$, to the defining equation of the grid, $x_i^{n+1} - x_i^n = 0$. Here and in the following, an upper index indicates the time level and a lower index the spatial grid point. The action of the Galilean transformation (1) on this grid equation yields $\tilde{x}_i^{n+1} - \tilde{x}_i^n = x_i^{n+1} - x_i^n + \varepsilon(t^{n+1} - t^n)$, which clearly fails to be invariant for $\varepsilon \neq 0$. Here we assumed that all the grid points are defined on the same time layer, i.e. $t_{i+1}^n = t_i^n = t^n$. It can be checked that this assumption does not violate the invariance of most of the equations of hydrodynamics, see also [11] for more details.

Unfortunately, to maintain Galilean invariance it is also not sufficient to carry out the numerical simulations with a standard finite difference scheme in a constantly moving reference frame as proposed in [1]. It can be verified numerically that the resulting numerical solutions in the resting and in the convecting reference frames do not coincide, which is explicitly shown in Fig. 1 for a forward in time, centered in space (FTCS) discretization of Burgers equation. In this figure, we display the numerical solution at $t = 0.5$ in the resting reference frame (solid line) and in a reference frame which moves with constant velocity $\varepsilon_3 = 1$ (solid line with triangles) as in [1].

Instead of using a non-invariant finite difference scheme in a convecting reference frame, it is therefore desirable to construct proper finite difference discretizations that preserve the invariance group of a physical differential equations. The above observation on the incompatibility of stationary meshes with Galilean invariance have severe consequences on the design of finite difference models for the equations of fluid dynamics. In fact, it renders necessary to come up with strategies to combine the requirement of using moving meshes (in order to preserve Galilean invariance) with approaches that lead to discretization schemes having good numerical properties, such as stability, optimal grid adaptation (e.g. equidistribution of the discretization error) and the possibility for a parallel implementation. From a more general point of view, it is necessary to bridge the fields of group analysis and numerical analysis of differential equations.

To outline this connection for the example of Burgers equation considered in [1] is the main aim of the present paper. In Section 2 we discuss invariant finite difference schemes for Burgers equation. We construct three different types of invariant numerical schemes, namely Lagrangian discretizations, invariant adaptive Eulerian schemes and invariant schemes employing an evolution–projection strategy. We relate these schemes to the remedy for reducing the effect of violation of Galilean invariance proposed in [1]. Numerical results for the different schemes discussed are presented in Section 3. The final Section 4 contains the conclusions of the paper.

2. Invariant finite difference schemes for Burgers equations

As in [1], we introduce Burgers equation as a canonical example for high Reynolds number flows,

$$u_t + uu_x - \nu u_{xx} = 0, \tag{2}$$

where $\nu > 0$ is the viscosity, which could be scaled to 1 by means of an equivalence transformation. It is one of the most investigated models in the group analysis of differential equations, see e.g. [4,5,17]. Its maximal Lie invariance algebra $\mathfrak{g}$ is spanned by the basis elements

$$\partial_t, \quad \partial_x, \quad t\partial_x + \partial_u, \quad 2t\partial_t + x\partial_x - u\partial_u, \quad t^2\partial_t + tx\partial_x + (x - tu)\partial_u. \tag{3}$$