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An adaptive meshfree diffusion wavelet method for partial differential equations on the sphere

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ABSTRACT

An adaptive meshfree diffusion wavelet method for solving partial differential equations (PDEs) on the sphere is developed. Approximation formulae for Laplacian–Beltrami (∇^2) and gradient ($\vec{\nabla}$) operators are derived using radial basis functions (RBFs), and the convergence of these approximations to ∇^2 and $\vec{\nabla}$ is verified for two test functions. The matrix approximating the Laplace–Beltrami operator is used for the construction of the diffusion wavelet. The diffusion wavelet is used for the adaptation of node arrangement as well as for the fast computation of dyadic powers of the matrices involved in the numerical solution of the PDE. The efficiency of the method is that the same operator is used for the construction of the diffusion wavelet and for the approximation of the differential operators. As a part of the wavelet method the behaviour of the compression error with respect to different parameters involved in the construction of the diffusion wavelet is tested. The CPU time taken by the proposed method is compared with the CPU time taken by the RBF based collocation method and it is observed that the proposed method performs better. The method is tested on three test problems namely spherical diffusion equation (linear), problem of computing a moving steep front (nonlinear) and problem of Turing patterns (system of nonlinear reaction–diffusion equations).

1. Introduction

Partial differential equations (PDEs) are widely used for realistic representation of real world problems such as fluctuations in stock markets, epidemiological models, climate modelling etc. In some situations the domain of these PDEs is a general manifold [2]. Many attractive mathematical properties of wavelets (namely efficient multiscale decompositions, compact support, vanishing moments, and the existence of fast wavelet transform etc.) in conjunction with the techniques for preconditioning and compression of operators and matrices, have motivated their use for numerical solutions of PDEs. Wavelet methods have been developed for most of the linear PDEs such as Laplace/Poisson equations [8] and advection diffusion problems [29]. Also for nonlinear PDEs, there exists a large spectrum of wavelet methods, which have been applied to Burger's equation [27,41], reaction–diffusion equations [18] and Stokes equation [11].

While solving a PDE numerically one can either choose to work on a static node arrangement [46] constructed at the beginning of the computation or can opt for an adaptive node arrangement which will keep on modifying itself according to the numerical solution of PDE at different times [32,3,23,1]. With a static node arrangement we need a large set of node points to discover all the features of the solution but this will increase the computational as well as storage cost.

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In some cases the set required for a static node arrangement to capture all the features of the solution may exceed the practical limitations. To deal with these problems we work on an adaptive node arrangement. In case of an adaptive node arrangement, instead of taking a larger set of node points, more node points are added only in the areas where the solution of PDE possesses sharp variations. Computational and storage costs will be saved by using an adaptive node arrangement.

In recent years many wavelet based adaptive numerical methods for solving PDEs were developed [27,22,6,33] on the manifolds with zero curvature. The critical issue is to solve PDEs on general manifolds. Many techniques have been developed to construct wavelets on general manifolds. For example:

1. In [5,9], wavelet bases are constructed on a special type of manifolds which can be represented as a disjoint union of smooth parametric images of a standard cube. The construction is based solely on smooth parametrization of the unit cube, which has several disadvantages from a practical point of view.
2. The above problem is resolved in [10], where wavelet basis functions are constructed from an initial finite element discretization of the domain.
3. In [17], spherical Shannon wavelets are constructed that form an orthogonal multiresolution analysis on the sphere.
4. The second generation wavelet is developed by Sweldens and co-workers in [37] and the main advantage of this wavelet is that it can be custom designed for general manifolds.

Despite of vast literature of wavelets on general manifolds, the wavelet theory for numerical solutions of PDEs on general manifolds is still in its nascent stage. The second generation wavelet led to wavelet based solutions of PDEs initially in [41], where one-dimensional Burger's equation, modified Burger's equation and laminar diffusion frame problems are solved. This paper was a major turning point for the people using wavelets for numerical solutions of PDEs because second generation wavelet has many practical advantages over the existing wavelets. This wavelet was later used to solve PDEs on the sphere in [28], where a dynamic adaptive numerical method for solving PDEs on the sphere is developed using second generation spherical wavelet. The main difficulty with second generation wavelet is that we require an initial mesh structure to approximate the manifold (e.g. sphere can be approximated using an icosahedron mesh). However, generating an initial mesh for an arbitrary manifold is difficult.

This difficulty can be handled with meshfree methods. Meshfree methods are formulated based on a set of scattered nodes and mesh-related difficulties are avoided as no mesh is used. Hitherto, the meshfree methods based on wavelets are very less developed and to best of our knowledge, the developed wavelet-meshfree methods are limited to flat geometry [21,45,43]. In this paper, we develop an adaptive meshfree diffusion wavelet method on the sphere (AMDWMS).

The diffusion wavelet was introduced by Coifman and his co-workers in 2006 [7]. The important features of this wavelet are that it can be constructed on general manifolds and its construction does not require an initial mesh discretizing the manifold. These features make diffusion wavelet a suitable choice for meshfree methods on general manifolds.

The AMDWMS developed in this paper can solve PDEs on an adaptive node arrangement on the sphere and can be generalised to general manifolds. AMDWMS will use RBFs for interpolation of functions and for approximation of differential operators. The theory of interpolation of continuous functions by RBFs is well understood [24,30,4,34]. In 1990, E.J. Kansa introduced the concept of solving PDEs using RBFs [25,26]. Since then RBFs are continuously used for solving PDEs on manifolds of zero curvature [16,15,12,38], but RBFs are not widely used to solve PDEs on general manifolds. To best of our knowledge the only work done is by Q.T.L. Gia [20,19], where Wendland's RBFs are used in approximation of elliptic and parabolic PDEs on the sphere. In their work, the initial conditions for the PDEs are chosen in such a way that the Laplace–Beltrami and gradient operators are reduced to a very simple form. Therefore approximation formulae for these operators are not derived in their work. In the present paper we have obtained the approximation formulae for Laplace–Beltrami and gradient operator on the unit sphere and unit cube (in most generalised form) using Wendland's RBFs.

The beauty of the proposed method is that the approximation of Laplace–Beltrami operator obtained using RBFs is used both for the construction of the diffusion wavelet as well as for the approximation of the differential operator involved in PDEs. The diffusion wavelet is used for adapting node arrangement and for the fast computation of the dyadic powers of the matrices involved in the numerical solutions of the PDEs.

The technique in this paper somehow resembles with the technique used in [14], where PDEs on an interval with periodic boundary conditions are considered. In the review paper [35] the wavelet methods for solving PDEs are classified into following categories

1. Pure wavelet methods.
2. Adaptive multiresolution methods.
3. Lagrangian wavelet methods.
4. Space–time wavelet methods.
5. Wavelet optimized adaptive methods.
6. General manifold treatment.

The proposed method will obviously fit in the second category. Moreover, it will also fit in the last category as the method uses nontensorial wavelet viz. diffusion wavelet to solve the PDEs on a general manifold.