An energy-conserving Galerkin scheme for a class of nonlinear dispersive equations

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ABSTRACT

A Galerkin scheme is presented for a class of conservative nonlinear dispersive equations, such as the Camassa–Holm equation and the regularized long wave equation. The scheme has two advantageous features: first, it is conservative in that it keeps the discrete analogue of the continuous energy conservation property in the original equations; second, it can be formulated only with cheap H^1-elements even if the original equations include third derivative u_{xxx}. Numerical experiments confirm the stability and effectiveness of the proposed scheme.

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1. Introduction

In this paper numerical integration of a class of nonlinear dispersive equations:

$$u_t - u_{xxt} + \kappa u_x + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx}), \qquad x \in \mathbb{R}, \quad t > 0, \tag{1}$$

is considered. Eq. (1) describes a wide variety of nonlinear dispersive phenomena depending on the values of κ and γ. With $\kappa \geqslant 0$, $\gamma = 1$, it reduces to the Camassa–Holm equation (CH):

$$u_t - u_{xxt} + \kappa u_x = 2u_x u_{xx} + uu_{xxx} - 3uu_x, \tag{2}$$

which models unidirectional propagation of shallow water waves [6,7], with u representing the fluid velocity in the x direction (or equivalently the height of the fluid's free surface), and κ the critical shallow water wave speed. The CH has a bi-Hamiltonian structure, is completely integrable [18], and has global solutions [10]. It also has solitary waves, but they are in sharp

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contrast to, for example, those of the Korteweg–de Vries equation, in that the solitary waves become peaked in the limit of $\kappa \to 0$ (called "peakons"), which describe in physical context "wave-breaking" [28]. When $\kappa = 0, \gamma \in \mathbb{R}$, the equation reduces to the Dai equation [14]:

$$u_t - u_{xxt} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx}), \tag{3}$$

which describes propagation of finite-length and small-amplitude waves in cylindrical compressible hyperelastic rods. In this case u represents the radial stretch, and γ the material constant which, for example, ranges from -29.4760 to 3.4174 [15]. Although this equation looks quite similar to the CH, its solutions behave in completely different manners depending on γ [12,15,30]. For $\gamma < 1$, solitary waves of the Dai equation are smooth. When $\gamma = 1$ the equation reduces to the CH equation with $\kappa = 0$, and thus solitary waves become peaked. If γ exceeds one, the singularities in the solutions become even stronger such that at some points derivatives are no longer bounded ("cusped"); physically, it expresses the "rod-breaking." Another example of (1) is the so-called regularized long wave (or simply the BBM) equation [3]:

$$u_t - u_{xxt} = -3uu_x - u_x, \tag{4}$$

which is obtained by setting $\kappa = 1, \gamma = 0$. In contrast to the preceding two equations, all solutions of the BBM are global, and solitary waves are smooth.

Motivated by the physical and mathematical relevance of these PDEs, some effort have been already devoted to the numerical computation of the Eq. (1). Below are such examples. For the CH, several standard pseudospectral schemes [6,7,23], a finite-difference scheme [21], a specialized scheme using the multi-peakon structure of the equation [22], and a multi-symplectic scheme [9] have been studied (see also [1]). In the context of Galerkin schemes, quite recently a local discontinuous Galerkin scheme has been also proposed [29], where its high efficiency and stability have been revealed. For the BBM, we refer the readers to [16,20] and the references therein. For the Dai equation, we could not find any, which might be attributed to the fact that the equation itself is quite new (proposed in 1998).

The aim of the present paper is to give a new reliable numerical scheme for Eq. (1), from a different perspective from the above mentioned numerical studies. The key fact here is that the Eq. (1) has an invariant under appropriate boundary conditions. With numerical analysis in mind, let us choose the periodic boundary condition:

$$u(x,t) = u(x+L,t), \qquad x \in (-\infty, \infty), \quad t > 0. \tag{5}$$

Then we see the quantity

$$-\frac{1}{2}\int_0^L \left(\kappa u^2 + u^3 + \gamma u u_x^2\right)\mathrm{d}x \tag{6}$$

is strictly preserved along the solution (see Theorem 1). It often corresponds to some physical energies of fluids or rods, and thus is called the "energy." In the present paper, in the purpose of constructing a "reliable" numerical scheme, we demand our numerical scheme to keep this energy conservation property in discrete setting. In recent years, such "energy-conserving" numerical schemes have drawn much interest and been extensively studied for various PDEs, since they are more likely to give stabler and qualitatively better computations [4,19,26] (see also [24] for nearly-conservative method for Hamiltonian PDEs). To the best of the authors' knowledge, however, so far strictly conserving scheme for (1) has not yet been published in the literature (except a master's thesis by Takeya [27] which considered conserving finite-difference schemes for the CH, and a study on a nearly-conservative multi-symplectic scheme for the CH [9]). Our main idea for constructing a strictly-conservative scheme is to utilize the concept of "discrete partial derivatives", which has been introduced in [25] by one of the present authors for designing conservative (or dissipative, respectively) Galerkin schemes for certain conservative (dissipative) PDEs in variational form. It will be shown that with some trick Eq. (1) can be also written in variational form, and the idea and tools above can be utilized for (1).

In this mission, we further like to demand that our numerical scheme is able to be formulated within the space H^1, the standard first-order Sobolev space, due to the following two reasons. First, simply from the computational perspective, we hope to keep the possibility of utilizing cheap H^1-elements (instead of relatively expensive C^1-elements). Second and more importantly, since the solutions of the target Eq. (1) can develop derivative singularities (e.g. peakons), it seems natural to work within the space H^1 rather than H^3 which is seemingly required to treat the third-order Eq. (1). This point seems to have not been explicitly emphasized in the existing numerical studies, where more or less stronger regularity is implicitly assumed by using some standard finite-difference or pseudospectral discretization.

With such a goal fixed—an energy-conserving scheme implementable within H^1—we face a big difficulty: as far as the authors know, there has been no H^1-formulation of Eq. (1) in the literature that directly gives rise to the energy conservation property. Although some H^1-formulation have been found so far in order to justify peakons (see Section 2), the energy conservation property of their H^1 solutions can be proved only in so indirect manners that they cannot be followed in discrete setting. As a solution to this issue, in this paper we present a new H^1-formulation of the problem from which the energy conservation property can be quite easily and directly derived.

The rest of the present paper is organized as follows: In Section 2, mathematical preliminaries regarding the target Eq. (1) are summarized. Then in Section 3, the proposed scheme is presented, and its properties are discussed. Its application examples are shown in Section 4 with various numerical experiments that illustrate the effectiveness of the proposed scheme.