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Reduced basis method for multi-parameter-dependent steady Navier–Stokes equations: Applications to natural convection in a cavity

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ABSTRACT

This work focuses on the approximation of parametric steady Navier–Stokes equations by the reduced basis method. For a particular instance of the parameters under consideration, we are able to solve the underlying partial differential equations, compute an output, and give sharp error bounds. The computations are split into an offline part, where the values of the parameters are not yet identified, but only given within a range of interest, and an online part, where the problem is solved for an instance of the parameters. The offline part is expensive and is used to build a *reduced* basis and prepare all the ingredients – mainly matrix–vector and scalar products, but also eigenvalue computations – necessary for the online part, which is fast.

We provide a model problem – describing natural convection phenomena in a laterally heated cavity – characterized by three parameters: Grashof and Prandtl numbers and the aspect ratio of the cavity. We show the feasibility and efficiency of the *a posteriori* error estimation by the natural norm approach considering several test cases by varying two different parameters. The gain in terms of CPU time with respect to a parallel finite element approximation is of three magnitude orders with an acceptable – indeed less than 0.1% – error on the selected outputs.

1. Introduction and motivation

The reduced basis approximation (see [23,30,21]) is a discretization method for the solution of parametrized partial differential equations. It permits *rapid* and *reliable* evaluation of input–output relationships *in the limit of many queries* – in the design, optimization, control and characterization contexts. We describe here a non-linear example: the case of the steady incompressible Navier–Stokes equations to model natural heat convection with more than one (physical, geometrical) parameter in affine dependence.[1] The case with one (physical) parameter within an affine parametric dependence has been treated in detail in [3] considering high Grashof numbers ($\sim 10^7$) and using a natural norm approach, and, previously, in [20,36] for lower Grashof numbers ($\sim 10^4$). A field of interest for this kind of applications deals with microfluidics, in biomedical sciences, environmental sciences and, more generally, with mechanical engineering (automotive and aerospace industry).

The use of the reduced basis method in numerical fluid dynamics is aimed at providing real-time solutions and information on fluid mechanics outputs. Its extension to steady Navier–Stokes equations, which requires treatment of non-linearities, provides, e.g., an efficient optimization toolbox in design problems with a certain degree of complexity. The study of parametrized systems is well suited also to carry out shape optimization and shape design problems considering several geometrical parameters.

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[1] Affine dependence is defined by Eq. (5) and roughly means that the parameters enter in the weak form as a coefficient in front of the integrals.

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The reduced basis approach and associated *offline–online* procedures can be applied without serious computational difficulties to quadratic non-linearities. Much work focuses on the stationary incompressible (quadratically non-linear) Navier–Stokes equations [6,8,10]: suitable stable approximations were first considered in [10,12,22], and more recently [24,28] also for non-affine parametric dependence; rigorous *a posteriori* error estimation – within the general Brezzi–Rappaz–Raviart ("BRR") *a posteriori* framework [1,2] – is considered in [3,20,36,37]. The latter is admittedly quite complicated, and presently limited to few parameters – Reynolds, Prandtl, Grashof numbers or an aspect ratio, as an example of geometrical parametrization. In this work we follow this line and focus our attention on the following aspects: (i) the efficient treatment of the non-linear term; (ii) the geometrical and physical parametrization; (iii) the incorporation of a stable approximation for pressure [31]; (iv) an accurate and feasible *a posteriori* error estimation based on the natural norm approach [3].

The aim of this paper is to provide extensive tests to validate and generalize the reduced basis method for natural convection problems in a rectangular cavity (see Fig. 1) increasing the number of physical and geometrical parameters: in addition to the Grashof number (Gr), used in [3], we consider, the Prandtl number (Pr) and the aspect ratio (A) of the laterally heated cavity, combined only in couples such as Gr–Pr, Gr–A, Pr–A. The combination of physical and geometrical parameters in the same problem provides a wide variety of applications involving thermo-fluid-dynamics.

An additional effort has been devoted in testing all the ingredients we need to compute error bounds and *a posteriori* error estimation (as certificate of fidelity of the methodology) in the multi-parametric case: in particular the lower bound to the inf–sup constant and the eigenvalues problems involved [1–3]. In the end, given a parameter value, either we are able to give an explicit correct error bound, or we cannot ensure existence or uniqueness of the solution and then of the output (which means that we have to enrich our basis).

The present work is organized as follows: after this introduction, as a short review on reduced basis for Navier–Stokes equations, in Section 2.1 the natural convection problem is presented (see Fig. 1). In Section 2.3 we recall the reduced basis formulation for Navier–Stokes equations, then in Sections 2.4–2.6 we recall all the principal ingredients for the *a posteriori* error estimation based on natural norm and existence and uniqueness results. The stabilization of the reduced basis and the offline and online algorithms are considered in Section 3, with references to the literature for technicalities [31,21,30]. In Section 4 numerical results and computational costs and savings are reported for three different parameter combinations. Some conclusions and description of future work follow in Section 5.

1.1. Offline–online computational decomposition

One of the keys in the reduced basis method is the decomposition of the computational work into an *offline* and an *online* stage (see [30,21]).

The former is carried out independently from a specific parameter of a problem at hand. A greedy algorithm is performed to search for the parameters that provide a heuristically optimized *reduced* basis for the Galerkin approximation, see [30] for comparison with other methodologies and performances. In the meantime, the ingredients for the resolution of the reduced discrete system and for the computation of the dual norm of the residual are computed. These hang on finite element matrix–vector and vector–vector products. In a second stage the error bound ingredients are built by solving a set of generalized eigenvalue problems.

The *online* stage involves a (some) particular instance(s) of the parameter. The system that has to be solved is the Galerkin projection on the space spanned by the reduced basis. The complexity of both the resolution of this system and the computation of the error bounds depends on the chosen number of basis functions but is independent from the number of degrees of freedom of the underlying finite element problem.

1.2. Abstract formulation

We are interested in the numerical approximation of parameter ($\boldsymbol{\mu}$) dependent non-linear partial differential equations and the prediction of an "output of interest" which is a functional of the field variable $\mathbf{u}^e(\boldsymbol{\mu})$,

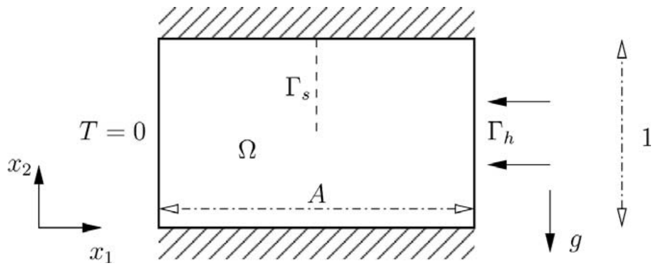

Fig. 1. A closed cavity. On the left the temperature is constant, on the right the heat flux is constant, and the top and the bottom of the cavity are insulated.