Journal of Computational Physics 229 (2010) 8281–8295

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

RBF-FD formulas and convergence properties

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ARTICLE INFO

Article history:
Received 23 March 2010
Received in revised form 6 July 2010
Accepted 7 July 2010
Available online 29 July 2010

Keywords:
Radial basis functions
Mesh-free

ABSTRACT

The local RBF is becoming increasingly popular as an alternative to the global version that suffers from ill-conditioning. In this paper, we study analytically the convergence behavior of the local RBF method as a function of the number of nodes employed in the scheme, the nodal distance, and the shape parameter. We derive exact formulas for the first and second derivatives in one dimension, and for the Laplacian in two dimensions. Using these formulas we compute Taylor expansions for the error. From this analysis, we find that there is an optimal value of the shape parameter for which the error is minimum. This optimal parameter is independent of the nodal distance. Our theoretical results are corroborated by numerical experiments.

1. Introduction

Radial basis functions (RBF) originate as a very efficient technique for interpolation of multidimensional scattered data (see [8] and references therein). Later, it became popular as a truly *mesh-free* method for the solution of partial differential equations (PDEs) on irregular domains. This application of RBFs was first proposed by Kansa [14,15] and it is based on enforcing collocation of the PDE in a set of scattered nodes, to compute a global solution in the space spanned by a set of identical RBFs translated to a set of RBF centers. The main advantages of the method are ease of programming and potential spectral accuracy, but its main drawback is ill-conditioning of the resulting linear system. To overcome this drawback a *local* version of the method was later proposed by several authors [3,24,26] simultaneously. The idea of the *local RBF* method, is to sacrifice the spectral accuracy inherent to the *global* method, in order to have a sparse better-conditioned linear system capable of solving large multidimensional PDEs. Another advantage of the local version of the method is its suitability for problems with discontinuous boundary conditions [1,5].

The local RBF method can also be considered as a generalization of the classical finite difference (FD) method to scattered node layouts. In classical finite differences, derivatives of a function u at a given point are approximated as linear combinations of the values of u at some surrounding nodes. In 1-D, for example, the kth-derivative at node x_j is approximated by

$$\left.\frac{d^k u}{dx^k}\right|_{x=x_j} \approx \sum_{i=1}^{N} w_{j,i}^{(k)} u(x_i) \quad j = 1, \ldots, N,$$

where x_i is a set of surrounding nodes which usually are equispaced. The unknown weights $w_{j,i}^{(k)}$ are usually computed using polynomial interpolation [9]. These 1-D formulas can be combined to create FD formulas for partial derivatives in two or more dimensions, provided that the nodes in the stencil are located on some kind of structured grid, which severely limits

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doi:10.1016/j.jcp.2010.07.008

the geometric flexibility of the method. In the case of RBF finite difference formulas (*RBF-FD*) this restriction is eliminated since the weights are obtained by RBF interpolation on the set of surrounding nodes.

Once the weights for the derivatives appearing in the PDE have been determined for each scattered node, the differential operator is enforced at each of those nodes. This procedure leads to a sparse, linear system of equations whose solution yields the approximate values of u at the nodes. This local RBF method has been successfully applied to solve a variety of problems [1,4,5,18,22,24,25].

However, papers addressing the convergence properties of the method are more scarce. It is well known that the local method lacks the spectral accuracy of the global RBF method, but the exact dependence of the error with average distance between nodes h, shape parameter c, and number of supporting nodes N, is not known. We mention, though, that Ding et al. [6] carried out numerical experiments using Poisson's equation on an equispaced grid to experimentally determine these dependencies. They found an error estimate $\epsilon \approx O((h/c)^n)$ in which n is a constant dependent on the number of nodes N used in the formulas ($n \approx 1.9$ for $6 \leqslant N \leqslant 9$, $n \approx 3.6$ for $9 < N \leqslant 27$, $n \approx 4.9$ for $27 < N \leqslant 34$).

Fornberg and coworkers [7,10] analyzed the behavior of RBF interpolants in the limit of increasingly flat radial functions ($c \to \infty$). They found that in the 1-D case, with very simple requirements on the basis functions, the interpolants converge to the Lagrange interpolating polynomial and, therefore, in this limit *RBF-FD* differentiation is equivalent to the standard finite difference method. Wright and Fornberg [27] used Hermite RBF interpolation method to derive new finite difference formulas (*RBF-HFD*) which also include a linear combination of derivatives at some surrounding nodes. They used cardinal RBF interpolants to derive *RBF-FD* and *HFD* formulas in some simple cases and studied their behavior in the limit of flat basis functions. They also analyzed numerically the dependence of the error on the shape parameter by using them to solve some simple elliptic PDE problems.

In this work we address the convergence properties of *RBF-FD* formulas on equispaced and non-equispaced grids and analyze the dependence of the error with nodal distance h, shape parameter c, and number of supporting nodes N. The main result of our study is to analytically show the existence of an optimal value of the shape parameter that minimizes the truncation error. The optimal value is independent of the nodal distance and only depends on the value of the function and its derivatives.

The paper is organized as follows: In Section 2 we describe the RBF-FD formulas and how to determine the unknown weighting coefficients. In Section 3 we use Taylor series expansion in the limit $c \gg h$ to derive closed form expressions of the weighting coefficients for first and second order derivatives. A series expansion in powers of h leads to closed form expressions for the error as a function of h and c. In Section 4 we derive the corresponding expressions for the error of RBF-FD formulas to approximate the Laplacian. The results of Sections 3 and 4 are used in Section 5 to derive the optimal value of the shape parameter. Section 6 extends these results to the case of non-equispaced nodes. Finally, we summarize the main results of this work in Section 7.

2. RBF-FD formulation

In this section we describe how the RBF-FD formulas are derived and how the weights can be exactly computed. Consider a stencil consisting of N scattered nodes $\mathbf{x}_1,\ldots,\mathbf{x}_N$, and a differential operator $\mathcal{L}$. For a given node, say $\mathbf{x}_1$, the objective is to approximate $\mathcal{L}u(\mathbf{x}_1)$ as a linear combination of the values of u at the N scattered nodes, so that

$$\mathcal{L}u(\mathbf{x}_1) \approx \sum_{i=1}^{N} \alpha_i u(\mathbf{x}_i). \tag{1}$$

To determine the weighting coefficients α_i, a set of base functions $\phi_i(\mathbf{x})$, $i = 1,\ldots,N$ are required. In that base,

$$\mathcal{L}\phi_j(\mathbf{x}_1) = \sum_{i=1}^{N} \alpha_i \phi_j(\mathbf{x}_i), \quad j = 1,2,\ldots,N. \tag{2}$$

This is a system of N linear equations on N unknowns whose solution yields the unknown weighting coefficients α_i. In the following we will use multiquadrics as RBFs,

$$\phi_i(\mathbf{x}) = \sqrt{c^2 + \|\mathbf{x} - \mathbf{x}_i\|_2^2},$$

where c is the shape parameter. As c increases the multiquadrics becomes increasingly flat and this has an important effect in the accuracy of the approximation. The general behavior is such that the larger the shape parameter c, the smaller the approximation error. However, the multiquadric RBF approximation suffers from a trade-off principle [21], i.e. increasing the shape parameter to improve the accuracy results in a more ill-conditioned matrix and, therefore, to a significant increase of rounding errors.

3. One-dimensional RBF-FD Formulas

In this section we show how to derive the exact RBF-FD formulas for first and second derivatives. We compute the limit of these formulas for $c \gg h$, and perform a Taylor expansion of the error in powers of h.