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# Optimization of spectral functions of Dirichlet–Laplacian eigenvalues

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ABSTRACT

We consider the shape optimization of spectral functions of Dirichlet–Laplacian eigenvalues over the set of star-shaped, symmetric, bounded planar regions with smooth boundary. The regions are represented using Fourier-cosine coefficients and the optimization problem is solved numerically using a quasi-Newton method. The method is applied to maximizing two particular nonsmooth spectral functions: the ratio of the $n$th to first eigenvalues and the ratio of the $n$th eigenvalue gap to first eigenvalue, both of which are generalizations of the Payne–Pólya–Weinberger ratio. The optimal values and attaining regions for $n \leqslant 13$ are presented and interpreted as a study of the range of the Dirichlet–Laplacian eigenvalues. For both spectral functions and each $n$, the optimal attaining region has multiplicity two $n$th eigenvalue.



## 1. Introduction

Denote by $\mathcal{D}$ the set of star-shaped, symmetric, bounded planar regions with smooth boundary. The Dirichlet–Laplace (D–L) eigenvalue problem for a region $D \in \mathcal{D}$ seeks eigenvalues $\lambda \in \mathbb{R}$ and eigenfunctions $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$, nontrivial, such that

$$-\Delta u = \lambda u, \quad \mathbf{x} \in D \tag{1a}$$

$$\mu = 0, \quad \mathbf{x} \in \partial D \tag{1b}$$

There is a tremendous body of work studying the distribution of the D–L eigenvalues and the properties of D–L eigenfunctions—see, for example, [1–5] and references within. Notably, there are a countable number of positive eigenvalues with no finite accumulation point. These eigenvalues are invariant under isometry of the domain (rotation and translation) and satisfy domain monotonicity (i.e. larger regions have smaller eigenvalues: $D \subset D' \Rightarrow \lambda'_k \leqslant \lambda_k$). Both of these facts are consequences of the max–min principle, stated

$$\lambda_k = \max_{\{v_j\}_{j=1}^{k-1}} \min_{\mathcal{Z}_{k-1}} \frac{\int_D |\nabla v|^2 \, d\mathbf{x}}{\int_D v^2 \, d\mathbf{x}} \tag{2}$$

where $\mathcal{Z}_{k-1} \equiv \left\{ v \in H_0^1(\Omega) \setminus \{0\} : v \perp \{v_j\}_{j=1}^{k-1} \right\}$. The ratio in Eq. (2) is called the Rayleigh quotient. Low-lying eigenvalues satisfy numerous isoperimetric or universal inequalities, a few of which are discussed in Section 2. The distribution of D–L eigenvalues for large $n$ satisfies Weyl's Law

$$\lambda_n(D) \sim 4\pi n A(D)^{-1} + o(n) \tag{3}$$

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where $A(D)$ is the area of $D \in \mathcal{D}$ [5,6,1]. Each eigenfunction is smooth ($C^\infty$) on $D$ and zero on a set of $C^\infty$ curves referred to as *nodal lines* with well-known properties. Closed form expressions for the eigenfunctions cannot generally be obtained unless the domain can be transformed into a separable coordinate system. If the domain has symmetry, the eigenfunctions are either even or odd with respect to the axis of symmetry, simplifying their computation. The D–L eigenvalue problem arises in a number of physical, engineering, and mathematical contexts including the study of vibrating membranes, electromagnetism, acoustic wave propagation, heat flow, the semi-classical approximation of quantum bound states, and number theory.

We denote by $\Lambda_n(D)$ the first $n$ increasingly-ordered D–L eigenvalues of a domain $D \in \mathcal{D}$ counting multiplicity and refer to the mapping $\Lambda_n : \mathcal{D} \to \mathbb{R}^n$ as the *D–L eigenvalue operator*. In this article, we study optimization problems of the form

$$\max_{D \in \mathcal{D}} \; F \circ \Lambda_n(D) \tag{4}$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is invariant to permutation of its arguments, i.e. $F(\mathbf{x}) = F(\pi(\mathbf{x}))$ where $\pi(\mathbf{x})$ is a reordering of the components of $\mathbf{x} \in \mathbb{R}^n$. For such $F$, the composition $F \circ \Lambda_n$ is referred to as a *spectral function* [7, Section 5.2]. Eq. (4) is a constrained shape optimization problem where the constraints are given by $n$ D–L eigenvalue equations. We generally refer to problem (4) as an *eigensystem-constrained shape optimization problem*.

One may interpret Eq. (4) as a method to study the range of the D–L eigenvalue operator. To this end, we consider two particular nonsmooth functions for $F(\mathbf{x})$, given by

$$R_n(\mathbf{x}) = \frac{[\mathbf{x}]_n}{[\mathbf{x}]_1} \tag{5}$$

$$G_n(\mathbf{x}) = \frac{[\mathbf{x}]_n - [\mathbf{x}]_{n-1}}{[\mathbf{x}]_1} \tag{6}$$

where $[\mathbf{x}]_p$ denotes the $p$th smallest component of $\mathbf{x} \in \mathbb{R}^n$. The spectral function $r_n(D) \equiv R_n \circ \Lambda_n(D)$ is the ratio of the $n$th to first D–L eigenvalues of the domain $D \in \mathcal{D}$. The spectral function $g_n(D) \equiv G_n \circ \Lambda_n(D)$ measures the gap between $r_n(D)$ and $r_{n-1}(D)$. Both $r_n(D)$ and $g_n(D)$ are invariant to translation, rotation, and dilation of the region $D \in \mathcal{D}$, so no additional constraints need be imposed in Eq. (4).

*Results and outline.* Our findings can be summarized:

1. In Sections 3–5, a BFGS quasi-Newton method is developed to solve the general eigensystem-constrained optimization problem in Eq. (4). In Section 3, we discuss the representation of the domain $D \in \mathcal{D}$ by Fourier-cosine coefficients, $\{b_k\}_{k=1}^{\infty}$ and a finite-dimensional approximation to $\mathcal{D}$, denoted $\mathcal{D}_m$. In Section 4, we compute the gradient of the objective function with respect to the Fourier-cosine coefficients, $b_k$. Then in Section 5, we discuss a numerical implementation of the method.
2. In Sections 6 and 7, the method is applied to the objective functions in Eqs. (5) and (6) for $n = 2, \ldots, 13$. The optimal values are given in Table 2 and the achieving regions are plotted in Figs. 3 and 4. For all $n$ considered, the domain $D_n^*$ maximizing either $r_n(D)$ or $g_n(D)$ has eigenvalues satisfying $\lambda_n(D_n^*) = \lambda_{n+1}(D_n^*)$. The results for both objective functions extend and support earlier work on the Payne–Pólya–Weinberger inequality and an eigenvalue multiplicity conjecture by Ashbaugh and Benguria [8].

## 2. Background and related work

Two well-written and extensive recent manuscripts on isoperimetric inequalities involving D–L eigenvalues can be found in [3,4]. The oldest and best-known such inequality is the Rayleigh–Faber–Krahn inequality, originally conjectured by Lord Rayleigh in 1894, stating that $\min \lambda_1(D)$ over the set of all membranes of fixed area is attained only by the disk.

The problems of maximizing $r_n(D)$ for $n = 2,3,4$ have been considered by many authors [3,4]. In 1955, Payne, Pólya, and Weinberger (PPW) showed that $r_2(D) \leqslant 3$ for all smooth bounded domains and correctly conjectured that the optimal value is attained by the disk [9]. This bound was studied numerically [10] and improved many times until finally being proved in 1992 by Ashbaugh and Benguria (AB) [11] and the corresponding inequality now bears the PPW name. With this proof, AB established that for the region $D^*$ (=disk) attaining the optimal value $r_2^* = \max r_2(D) \approx 2.539$, we have $\lambda_2(D^*) = \lambda_3(D^*)$. Subsequently, the range of the first two D–L eigenvalues has been studied numerically [12] and analytically (see [4, Section 6.4]). In 2003, after numerically searching through 65,000 trial and error regions, Levitin and Yagudin (LY) conjectured that $r_3^* \lesssim 3.202$ [13]. For the dumbell-shaped region $D^*$ with largest value $r_3$, they found $\lambda_3(D^*) = \lambda_4(D^*)$, supporting an earlier conjecture of AB [8]. In 1993, AB gave a bound for $n = 4$ stated $r_4^* \leqslant (r_2^*)^2 \approx 6.445$ [14].

Recently, there has been much work on the value of $r_n$ for larger values of $n$. Cheng and Yang have shown that

$$r_{n+1} \leqslant \frac{\sqrt{41}}{3} n \approx 2.134n \tag{7}$$

for $n \geqslant 3$ [15] and Harrell and Hermi have shown $r_{n+1} \leqslant \frac{21}{8} n = 2.625n$ [16]. Taking $n = 3$, we have: $r_4^* \lesssim 6.402$, a slight improvement over the bound given by AB. From Weyl's Law (3) and the Rayleigh–Faber–Krahn inequality, we expect that asymptotically