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A new smoothness indicator for improving the weighted essentially non-oscillatory scheme

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ABSTRACT

In this work, a new smoothness indicator that measures the local smoothness of a function in a stencil is introduced. The new local smoothness indicator is defined based on the Lagrangian interpolation polynomial and has a more succinct form compared with the classical one proposed by Jiang and Shu [12]. Furthermore, several global smoothness indicators with truncation errors of up to 8th-order are devised. With the new local and global smoothness indicators, the corresponding weighted essentially non-oscillatory (WENO) scheme can present the fifth order convergence in smooth regions, especially at critical points where the first and second derivatives vanish (but the third derivatives are not zero). Also, the use of higher order global smoothness indicators incurs less dissipation near the discontinuities of the solution. Numerical experiments are conducted to demonstrate the performance of the proposed scheme.

1. Introduction

High-order accurate weighted essentially non-oscillatory (WENO) schemes have recently been developed to solve the hyperbolic conservation law

$$u_t + f(u)_x = 0 \tag{1}$$

Based on the successful essentially non-oscillatory (ENO) scheme in [10], WENO schemes use the idea of adaptive stencils in the reconstruction procedure to automatically achieve high-order accuracy and non-oscillatory property near discontinuities [22,23]. The first WENO scheme was developed by Liu, Osher and Chan [14] for a third-order of accuracy and the classic fifth-order WENO scheme with a general framework for the design of the smoothness indicators and nonlinear weights was constructed in [12], which is referred to hereafter as WENO-JS. Later, very high order WENO schemes were developed in [1] and [8].

WENO schemes use a nonlinear convex combination of all the ENO candidate sub-stencils and assign each sub-stencil a weight between 0 and 1 based on local smoothness indicators. The basic weighting strategy is to use optimal weights for each of the lower order polynomials to combine an upwind scheme of maximum order in smooth regions of the solution, while assign small weights to those lower order polynomials whose underlying stencils contain discontinuities so that

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the ENO property is achieved. Although being *successful in a wide number of applications*, the *WENO methodology is* still in development to improve its convergence rate in smooth regions and *decrease the dissipation* near the discontinuities.

Henrick, Aslam and Powers pointed out that the actual convergence rate of the fifth-order WENO-JS was less than 5th for many problems [11]. They demonstrated that the smoothness indicator of WENO-JS failed to recover the maximum order of the scheme at critical points where the first or higher order derivatives of the function vanished. In the same paper, they proposed a new WENO-M scheme to disentangle this problem by modifying the smoothness indicator of WENO-JS with a mapping procedure to satisfy the sufficient criteria for fifth-order convergence. Compared to the WENO-JS scheme, the WENO-M scheme can achieve the optimal convergence order at critical points of smooth parts of the solution and reduce the numerical dissipation near discontinuities.

With a different weighting formulation, Borges, Carmona, Costa, and Don [5] introduced another version of the fifth-order WENO scheme (called WENO-Z) which uses a global higher order reference value for the smoothness indicators and drives the WENO weights to the optimal values faster than the WENO-M scheme. The WENO-Z presents less dissipation than WENO-JS but at the first order critical points the convergence order is 4th and will degrade to 2nd when higher order critical points are encountered. In [6], a closed-form formula for the WENO-Z scheme of higher than fifth-order accuracy was derived.

A new smoothness indicator for the WENO-Z scheme was recently developed in [9] and a new 6th-order global smoothness indicator was devised. The convergence rate of the resulted WENO scheme can recover the optimal fifth-order even at the first order critical point, while, it drops to a same lower rate as WENO-Z when a second order critical point is met.

Recently, developments have also been made in designing WENO schemes on unstructured meshes in more space dimensions. For example, a third-order finite volume WENO scheme was constructed on three-dimensional tetrahedral meshes [27] and an arbitrary high order finite volume WENO scheme on unstructured grids in two and three space dimensions was developed by using the ADER (ADER stands for Arbitrary Derivative Riemann problem) approach [7]. In [13] a third-order compact central WENO scheme was presented for multidimensional conservation laws.

Excepting applications for the hydrodynamics problems, the WENO schemes have also been applied for solving the magnetohydrodynamics (MHD) problems. For example, the divergence-free WENO schemes for simulating the MHD flows were designed by Balsara [2] and it was found that for MHD the ADER-WENO schemes are better than the Runge–Kutta (RK)-WENO schemes. In [3,4] efficient techniques for WENO interpolation have been proposed in developing the ADER-WENO schemes for the divergence-free MHD. They presented very elegant and compact formulations of WENO reconstruction as well as their implementation details by expressing the interpolating functions in modal space of Hermite polynomial.

In this article, we introduce a new smoothness indicator η for evaluating the local smoothness of the numerical solution in a stencil. The new local smoothness indicator is calculated based on the Lagrangian interpolation polynomial and has a more succinct form compared with the classical one of the WENO-JS scheme. The WENO scheme that equipped with the new smoothness indicator η is designated as the WENO-η scheme. Observations from the given numerical experiments manifest that the WENO-η scheme has an equivalent performance as the WENO-JS scheme.

Furthermore, several higher order (up to 8th-order) global smoothness indicators τ are constructed, with which the associated WENO schemes (hereafter, denoted by WENO-Zη) can achieve fifth convergence order in smooth regions, even at the second order critical points where the first and second derivatives vanish. Also, it is found that the use of higher order global smoothness indicators incurs less dissipation near the discontinuities of the solution. Numerical experiments are presented to show that the WENO-Zη schemes provide at least similar or improved behaviors over the WENO-Z scheme.

The rest of this paper is organized as follows. In Section 2, a brief review of the WENO schemes for one-dimensional scalar conservation laws is provided. In Section 3, we present a new local smoothness indicator and several higher order global smoothness indicators for improving the WENO scheme. In Section 4, some numerical results are presented to show the capacity of the proposed WENO scheme. Section 5 concludes this paper.

2. Review of the WENO schemes

For the hyperbolic conservation law (1), the flux function $f(u)$ is split into two parts as $f(u) = f^+(u) + f^-(u)$ with $df^+(u)/du \geqslant 0$ and $df^-(u)/du \leqslant 0$. The semi-discretization form of (1) can be written as

$$\frac{du_i(t)}{dt} = -\frac{1}{\Delta x}(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}), \tag{2}$$

where the numerical flux is $h_{i+1/2} = h^+_{i+1/2} + h^-_{i+1/2}$. Hereinafter, only the positive part $h^+_{i+1/2}$ is described and the superscript "+" is dropped for brevity.

The flux of the classical fifth-order WENO-JS scheme is built through the convex combination of interpolated values $R_{i-2+l,3}(x_{i+1/2})$ $(l = 0, 1, 2)$

$$h_{i+\frac{1}{2}} = \sum_{l=0}^{2} \omega_l R_{i-2+l,3}(x_{i+\frac{1}{2}}) + O\left(\Delta x^5\right), \tag{3}$$

in which $R_{i-2+l,3}$ is the reconstruction polynomial on stencil $S_{i-2+l,3} = \{I_{i-2+l}, I_{i-1+l}, I_{i+l}\}$ $(I_i \equiv \{x_{i-1/2}, x_{i+1/2}\})$