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# Polymorphic nodal elements and their application in discontinuous Galerkin methods

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ABSTRACT

In this work, we discuss two different but related aspects of the development of efficient discontinuous Galerkin methods on hybrid element grids for the computational modeling of gas dynamics in complex geometries or with adapted grids. In the first part, a recursive construction of different nodal sets for *hp* finite elements is presented. They share the property that the nodes along the sides of the two-dimensional elements and along the edges of the three-dimensional elements are the Legendre–Gauss–Lobatto points. The different nodal elements are evaluated by computing the Lebesgue constants of the corresponding Vandermonde matrix. In the second part, these nodal elements are applied within the modal discontinuous Galerkin framework. We still use a modal based formulation, but introduce a nodal based integration technique to reduce computational cost in the spirit of pseudospectral methods. We illustrate the performance of the scheme on several large scale applications and discuss its use in a recently developed space-time expansion discontinuous Galerkin scheme.



## 1. Introduction

While discontinuous Galerkin (DG) methods were first proposed in the early 1970s in [36] it was not until the more recent development, initiated in the work of Cockburn and Shu [9,7,10–12], that these methods have matured into a powerful computational tool for the solution of systems of conservation laws and the equations of gas dynamics [4,13]. The extension to problems of viscous gas dynamics was initiated in [3,5] and this again has led to several related formulations [17,29,35] for the compressible Navier–Stokes equations. Many examples and further details along these lines can be found in [8,26,30].

In spite of these significant advances over the last decade, discontinuous Galerkin methods still suffer from being too expensive when compared to more traditional methods such as finite volume methods. This is particularly true for viscous

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problems, where the common solution approach is based on a *mixed* finite element formulation, which was introduced in [3] and extended to higher order problems in [40,41]. In recent developments for the DG discretization of second order terms [15,16,32], the introduction of auxiliary variables is circumvented by the use of *two* partial integrations, or by *multiple* partial integrations for higher order operators [6].

Apart from this, however, a major computational cost is found in the traditional use of full order integration in the basic implementation, leading to an excessive computational cost for nonlinear problems. Deriving inspiration from the classic spectral methods [22] it is natural to consider the use of a nodal basis, leading to a formulation which in spirit shares much with a spectral collocation formulation in which the boundary conditions are imposed weakly. Such methods, often known as spectral penalty methods, have been developed for the compressible Navier–Stokes equations in [18–20] and extended to non-tensorial elements in [21,23].

The main advantages of such a formulation are found in the exact reduction to the standard discontinuous Galerkin formulation for linear problems, hence ensuring the accuracy for smooth problems, and the quadrature free approach for nonlinear problems, leading to a dramatic reduction in the computational cost. Furthermore, the use of a nodal basis with the correct structure of the points along the edges and faces leads to a natural separation of the basis into boundary and internal degrees of freedom. This becomes particularly beneficial for schemes using a high-order basis. Although the loss of exact integration opens up for the possibility of instabilities driven by aliasing, this is a well known and well understood phenomenon within the community of spectral methods [22]. We shall return to this concern briefly later.

One of the limitations of past nodal based formulations and schemes is the reliance on either cubic or tetrahedral element shapes. While these suffice in many cases, for problems with significant geometric flexibility one is tempted to also use more general types of elements such as prisms and pyramids.

In this work, we explore how one constructs such general nodal elements, using a nodal recursive construction, and optimize these for maximum accuracy by minimizing the Lebesgue constant of the associated multivariate Lagrange polynomial. This is discussed in Section 2 and sets the stage for Section 3 where we discuss in detail the use of these elements in a discontinuous Galerkin scheme and return to the issues of aliasing and the potential for instabilities caused by this. We shall also discuss how nodal elements can be used with advantage in an already existing scheme based on a modal expansion and finally we use a recently developed explicit space-time discretization to arrive at the fully discrete explicit scheme. In Section 4 we demonstrate how this general scheme, employing polymorphic elements and local time-stepping, can be used with benefit for both linear and nonlinear wave problems and, finally, the full three-dimensional compressible Navier–Stokes equations. Most of the tests illustrate the potential for a four fold reduction in computational time without impacting the accuracy by using the nodal based approach for large scale simulations. Section 5 concludes with a few general remarks and outlook toward future work.

## 2. The nodal elements

We will first focus on defining different sets of high-order basis functions for a given grid cell $Q \subset \mathbb{R}^d$. We introduce the *monomial* basis $\{\pi_i\}_{i=1,\dots,N}$ for the space of polynomials with degree less than or equal than $p$, where every basis function $\pi_i$ is written as

$$\pi_i(\vec{x}) = x_1^{\alpha_1^i} \cdots x_d^{\alpha_d^i} \quad \text{with} \quad 0 \leqslant \alpha_1^i + \cdots + \alpha_d^i \leqslant p. \tag{1}$$

The dimension $N$ of this space depends on the order $p$ and on the spatial dimension $d$ of the grid cell $Q$ and is given by

$$N = N(p,d) = \frac{(p+d)!}{d!p!}. \tag{2}$$

Based on a monomial basis $\{\pi_i\}_{i=1,\dots,N}$ expanded in the barycenter of the grid cell $Q$ and its geometry the construction of an orthonormal basis $\{\varphi_i\}_{i=1,\dots,N}$ using Gram–Schmidt orthogonalization is straight forward. This basis set is characterized by the property

$$\int_Q \varphi_i(\vec{x})\varphi_j(\vec{x})d\vec{x} = \delta_{ij}, \tag{3}$$

which holds for *arbitrary* grid cell shapes. With this *modal* basis we are now able to proceed to define a set of *nodal* basis functions. Given a set of interpolation points $\{\vec{\xi}_j\}_{j=1,\dots,M_l} \subset Q$, we can construct the nodal Lagrange basis $\{\psi_j\}_{j=1,\dots,M_l}$ and the nodal degrees of freedom $\tilde{u}$ defined by the conditions

$$\begin{aligned} &\psi_j(\vec{\xi}_i) = \delta_{ij}, \\ &u(\vec{x}) := \sum_{j=1}^{N} \hat{u}_j \varphi_j(\vec{x}) \overset{!}{=} \sum_{i=1}^{M_l} \tilde{u}_i \psi_i(\vec{x}). \end{aligned} \tag{4}$$

Combining these conditions yields the transformations

$$\underline{\underline{V}}\,\underline{\hat{u}} = \underline{\tilde{u}} \quad \text{and} \quad \underline{\underline{V}}^T \underline{\psi} = \underline{\phi}, \tag{5}$$