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Approximation of mono-dimensional hyperbolic systems: A lattice Boltzmann scheme as a relaxation method

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ABSTRACT

We focus on mono-dimensional hyperbolic systems approximated by a particular lattice Boltzmann scheme. The scheme is described in the framework of the multiple relaxation times method and stability conditions are given. An analysis is done to link the scheme with an explicit finite differences approximation of the relaxation method proposed by Jin and Xin. Several numerical illustrations are given for the transport equation, Burger's equation, the *p*-system, and full compressible Euler's system.

0. Introduction

The strength of the lattice Boltzmann schemes lies in their effectivity. They are intensively used in academic and industrial contexts for numerical simulations of fluid dynamics. Their links with the mesoscopic physics and in particular with the Boltzmann equation make these schemes especially well adapted to simulate fluid phenomena obtained by asymptotic limits from the kinetic theory. However, it is sometimes awkward to fix the several parameters of a lattice Boltzmann scheme in order to simulate a given equation, even if this equation is written into a conservative form: the conservation of the energy is classically a difficulty that can involve to use two different schemes coupled by a source term [1]. Other very particular schemes were proposed and investigated in order to simulate the full compressible Euler system, with substantial works on the equilibria [2–6].

In this contribution, a new lattice Boltzmann scheme is introduced in order to approximate any mono-dimensional hyperbolic conservative system, the intended target being the various equations of the fluid dynamics: many systems are written as conservation laws and the propagation of the waves is an essential property. In particular, the equations obtained by the kinetic theory of gases (as Euler's equations) are of that type [7]. The followed methodology is to treat separately the equations of the system by leaving aside the Boltzmann equation as much as possible. Usually, in order to increase the dimension of the system—that is the number of conservation equations—densities with larger velocities are introduced with two consequences: first, the lattice of the velocities is extended with the obvious difficulties concerning the boundary conditions; second, added new velocities deeply modifies the scheme so that all previous investigations have to be redone. The proposed scheme denoted by $\mathrm{D}_1\mathrm{Q}_2^n$ is built by duplicating for each of the n conserved moments the well-known and simplest lattice Boltzmann scheme: the $\mathrm{D}_1\mathrm{Q}_2$ (one spatial dimension and two discrete velocities). Therefore, the results on the scalar equation can be easily extended to the system of n equations. Moreover, as the boundary conditions are written

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on the densities in the framework of the lattice Boltzmann schemes, the decoupling of the density functions extremely simplifies the choices of the incoming densities on the boundaries to fit the boundary conditions on the moments.

In [8], Jin and Xin introduced the relaxation method to replace a non-linear hyperbolic system of dimension n by a linear hyperbolic system of dimension $2n$ with a stiff source term—called a relaxation term as it enforces the added moment to relax to the flux of the initial system. The convergence of this method when the relaxation term becomes dominant was investigated in [9,10]. Many publications deal then with numerical relaxation schemes [11–13]. Otherwise, Junk reinterprets the lattice Boltzmann method—in particular the $\mathsf{D}_2\mathsf{Q}_9$—as an explicit finite differences discretization of a relaxation formulation for the incompressible Navier–Stokes equation in the diffusive scaling [14]. In this paper, the proposed $\mathsf{D}_1\mathsf{Q}_2^n$ scheme is related to a particular discretization of the relaxation method: a splitting between the linear hyperbolic part treated with an explicit finite differences discretization (Lax–Friedrichs discretization) and the relaxation part treated with an explicit Euler solver.

The first section of this paper is devoted to the scalar case: the $\mathsf{D}_1\mathsf{Q}_2$ scheme is written into the framework of d'Humières [15]; the equivalent equations are given up to the second order by using the Taylor expansion method [16,17]; the description of the scheme as a discretization of the relaxation method is then done and stability conditions are given; finally numerical illustrations for the transport equation and for Burger's equation are performed. In the second section, we consider the case of n-dimensional hyperbolic systems: the $\mathsf{D}_1\mathsf{Q}_2^n$ scheme is introduced and described; the Taylor expansion method is then used to obtain the second order equivalent equations and the link with the discretization of the relaxation method is done; finally numerical illustrations for the p-system and for the full compressible Euler equation are performed.

1. The $\mathsf{D}_1\mathsf{Q}_2$ scheme for the 1-D scalar equation

In this section, we consider the following mono-dimensional hyperbolic equation

$$\partial_t \mathrm{u}(t,x) + \partial_x \varphi(\mathrm{u})(t,x) = 0, \quad t > 0,\ x \in \mathbb{R}, \tag{1}$$

where the flux φ is a smooth function on $\mathbb{R}$. A two-velocities lattice Boltzmann scheme is used to approximate the solution of this equation.

1.1. Description of the scheme

We use the notation proposed by d'Humières in [15] by considering $\mathcal{L}$, a regular lattice in one dimension of space with typical mesh size Δx. The time step Δt is determined after the specification of the velocity scale λ by the relation:

$$\Delta t = \frac{\Delta x}{\lambda}. \tag{2}$$

For the scheme denoted by $\mathsf{D}_1\mathsf{Q}_2$, we introduce $\boldsymbol{v} = (-\lambda, \lambda)$ the set of the two velocities and we assume that for each node x of $\mathcal{L}$, and each v_j in $\boldsymbol{v}$, the point $x + v_j \Delta t$ is also a node of the lattice $\mathcal{L}$. The aim of the $\mathsf{D}_1\mathsf{Q}_2$ scheme is to compute a particles distribution vector $\boldsymbol{f} = (f_0, f_1)^{\mathrm{T}}$ on the lattice $\mathcal{L}$ at discrete values of time: it is a numerical scheme to approximately solve the PDEs

$$\partial_t f_j + v_j \cdot \nabla f_j = -\frac{1}{\tau_j}\left(f_j - f_j^{\mathrm{eq}}\right), \quad 0 \leqslant j \leqslant 1,$$

on a grid in space and time where f_j^{eq} describes the distribution f_j at the equilibrium and τ_j is the relaxation time (applied to f_j). The scheme splits into two phases for each time iteration: first, the relaxation phase that is local in space, and second, the transport phase for which an exact characteristic method is used.

The framework proposed by d'Humières [15] reduced here to the two moments denoted by $\boldsymbol{m} = (\mathrm{u}, \mathrm{v})^{\mathrm{T}}$ and defined for each space point $x \in \mathcal{L}$ and for each time t by

$$\mathrm{u} = f_0 + f_1, \qquad \mathrm{v} = \lambda(-f_0 + f_1). \tag{3}$$

The matrix of the moments $\boldsymbol{M}$ such that $\boldsymbol{m} = \boldsymbol{M}\boldsymbol{f}$ satisfies

$$\boldsymbol{M} = \begin{pmatrix} 1 & 1 \\ -\lambda & \lambda \end{pmatrix}, \qquad \boldsymbol{M}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2\lambda} \\ \frac{1}{2} & \frac{1}{2\lambda} \end{pmatrix}. \tag{4}$$

Let us now describe one time step of the scheme. The start point is the density vector $\boldsymbol{f}(x,t)$ in $x \in \mathcal{L}$ at time t, the moments are computed by

$$\boldsymbol{m}(x,t) = \boldsymbol{M}\boldsymbol{f}(x,t). \tag{5}$$

The relaxation phase then reads

$$\mathrm{u}^\star(x,t) = \mathrm{u}(x,t), \qquad \mathrm{v}^\star(x,t) = \mathrm{v}(x,t) + s\left(\mathrm{v}^{\mathrm{eq}}(x,t) - \mathrm{v}(x,t)\right), \tag{6}$$