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VIALS: An Eulerian tool based on total variation and the level set method for studying dynamical systems

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ABSTRACT

We propose a new Eulerian tool to study complicated dynamical systems based on the average growth in the surface area of a family of level surfaces represented implicitly by a level set function. Since this proposed quantity determines the temporal variation of the averaged surface area of all level surfaces, we name the quantity the *Variation of the Integral over Area of Level Surfaces* (VIALS). Numerically, all these infinitely many level surfaces are advected according to the given dynamics by solving one single linear advection equation. To develop a computationally efficient approach, we apply the coarea formula and rewrite the surface area integral as a simple integral relating the total variation (TV) of the level set function. The proposed method can be easily incorporated with a recent Eulerian algorithm for efficient computation of flow maps to speed up our approach. We will also prove that the proposed VIALS is closely related to the computation of the finite time Lyapunov exponent (FTLE) in the Lagrangian coherent structure (LCS) extraction. This connects our proposed Eulerian approach to widely used Lagrangian techniques for understanding complicated dynamical systems.

1. Introduction

It has long been an important task for many fields in science and technology to develop efficient tools to understand complicated dynamical systems. In this work, we develop simple numerical approaches to study a global property of a complex dynamical system. We consider the evolution of a given state in the extended phase space satisfying the ordinary differential equation (ODE)

$$\dot{\mathbf{x}}(t) = \mathbf{u}\big(\mathbf{x}(t), t\big) \tag{1}$$

with a given Lipschitz velocity field $\mathbf{u} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ and an initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

There are many existing tools to study dynamical systems. Coherent structure is one of these methods in understanding fluid motions. The goal is to segment the domain into different regions with similar behaviors according to a quantity such as the strain, kinetic energy, or the vorticity. Since these mentioned quantities are measured locally at fixed given locations and also instantaneously at fixed given times, their corresponding coherent structures are classified as Eulerian coherent structure (ECS). Another class of coherent structure is the Lagrangian coherent structure (LCS) which tries to partition the space–time domain into different regions according to a Lagrangian quantity moving along with passive tracers. Among many, a simple definition of LCS uses the finite-time Lyapunov exponent (FTLE) [17,15,16,42,21]. It measures the rate of

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separation between adjacent particles over a finite time interval with an infinitesimal perturbation in the initial location. For a given time, particles are first advected in the flow for a period of time to obtain the flow map which takes the initial particle location to its arrival location. Mathematically, we define the flow map $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ to be the mapping which takes the point $\mathbf{x}_0$ from $t = t_0$ to the particle location at the final time $t = t_0 + T$, i.e. $\Phi_{t_0}^{t_0+T}(\mathbf{x}_0) = \mathbf{x}(t_0 + T)$ with $\mathbf{x}(t)$ satisfying (1). Then the FTLE is computed using the norm of the Jacobian of this resulting flow map. In some recent works [22,23], we have proposed Eulerian methods to determine such Lagrangian quantity based on the level set method [35,41,34].

Ergodicity is another important concept in understanding dynamical systems [2,4]. Roughly speaking, the term ergodic in mathematics is used to describe a dynamical system in which the time-average of all system states (in the phase space) has similar behavior to their space-average. Following [30,31], [28,29] have developed a numerical approach to visualize the ergodic partition of discrete dynamical systems based on the ergodic partition theory. The algorithm first computes the time averages of various observables defined in the phase space, and then uses these averages to divide the phase space into a family of non-decomposable invariant sets in each of which all points are accessible from any initial takeoff location in the set. Roughly speaking, any particle in such an invariant set can arrive everywhere in the set at later times. To work on a continuous dynamical system, on the other hand, [45] has recently extended the theory of ergodic partition from [31] to analyze dynamics described by continuous-time nonlinear models such as fluid dynamics with time-dependent velocity profiles. In a recent paper [47], we have further extended that work by introducing a concept of coherent ergodic partition. We observe that for time-dependent flows, the partition will not necessarily give invariant sets in the phase space by simply computing the joint level sets of time averages of observables, and so the partition is in fact not ergodic. Having said that, all particles within the same partitioned component at a given time (all time averages are equal in the same component) should have very similar behaviors. This concept can still help us to better understand the structure of a dynamical system. For convenience, we will still call this partition an ergodic partition. In [45], the authors have studied the so-called coherent swing instability (CSI) phenomenon and have shown that the corresponding ergodic partition is uniformly bounded. In [47], we have generalized the result and have concluded that the coherent ergodic partition will form an invariant set in the extended phase space. Also, if the flow is periodic, the evolution of each ergodic component is periodic with the same period as the flow.

Yet the simplest approach to visualize the system, on the other hand, is still to propagate a set of Lagrangian particles to observe the evolution of any small scale feature in the flow. For example, developed based on the contour dynamics (CD) [48], the contour surgery method [7] simulated the evolution of a contour numerically represented by connected particles in two dimensions. The method has successfully introduced a surgery technique to avoid the exponential growth in the total length of a contour in complex flows. Extending the contour surgery method, [46] proposed a trajectory technique called contour advection with surgery which also incorporated a remeshing strategy to maintain a nice sampling rate by balancing the sampling distance with the local curvature of the contour. An efficient method for detection and prevention of self-intersection has been recently proposed in a series of careful studies [38–40]. Various other improvements and extensions can also be found in [8,11,9,10].

An interesting application of the contour advection algorithm is on the study of chaotic mixing in flow visualizations from atmospheric sciences, such as [32,33]. The idea is to quantify the level of the mixing by determining the growth rate of the length of a particular contour. If the growth of the length was exponential (which we will show that such condition might not be satisfied for all flows), the work introduced a corresponding Lyapunov exponent to measure the level of mixing. Mathematically, one computes $\lambda(T) = \frac{1}{T}\log(L(T)/L(0))$ where $L(t)$ is the length of the contour at time t. Numerically, the evolution of all Lagrangian particles are determined by a high order Runge–Kutta scheme while the velocity at their locations are computed by linearly interpolating the velocity field on mesh. The contour itself is represented using cubic spline between particles and therefore the total length can be easily approximated. It should be noted, however, that the choice of the initial contour might seriously affect the change in the total contour length at the later time. For example, we for simplicity consider a flow where the velocity field is periodic in time. If the initial contour is chosen to be the boundary of the coherent ergodic partition we have introduced in [47], the length of the contour will also be periodic in time. However, the situation will be different if the initial contour stays inside a connected component of the ergodic partition. Since each particle on the contour will reach every location in the set, the length of the contour might be unbounded above. A more detailed description will be given in Section 5.1.3.

In this paper, we first convert the Lagrangian-type contour advection algorithm to an Eulerian description by incorporating the level set method. We are definitely not the first one to point out the possibility of such an Eulerian version. In fact, one of the main purposes of the original level set method is to implicitly track the evolution of a codimension-one surface. In this work, we are not concentrated on the evolution of a single level set contour, nor even multiple curves. We consider a family of contours based on the level set method and propose an easily computed quantity named the *Variation of the Integral over Area of Level Surfaces* (VIALS) for quantifying the level of mixing for a dynamical system. Numerically, we propose to apply the coarea formula [13] to convert the summation of all surface areas into the total variation (TV) of the level set function which can then be easily approximated in our Eulerian framework.

Indeed, similar to the Lagrangian contour advection method, solutions from the VIALS might depend on the choice of the initial level set function. Rather than determining an initial condition to maximize the resulting VIALS, in this work we propose a finite set of n initial conditions for the level set function in $\mathbb{R}^n$ so that the VIALS for an arbitrary initial function can be computed based on one single interpolation. We will further prove that the VIALS is closely related to the computations of the finite time Lyapunov exponent (FTLE) for Lagrangian coherent structure (LCS) extraction. This provides