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Double sweep preconditioner for optimized Schwarz methods applied to the Helmholtz problem

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A R T I C L E I N F O

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A B S T R A C T

This paper presents a preconditioner for non-overlapping Schwarz methods applied to the Helmholtz problem. Starting from a simple analytic example, we show how such a preconditioner can be designed by approximating the inverse of the iteration operator for a layered partitioning of the domain. The preconditioner works by propagating information globally by concurrently sweeping in both directions over the subdomains, and can be interpreted as a coarse grid for the domain decomposition method. The resulting algorithm is shown to converge very fast, independently of the number of subdomains and frequency. The preconditioner has the advantage that, like the original Schwarz algorithm, it can be implemented as a matrix-free routine, with no additional preprocessing.

1. Introduction

Solving the Helmholtz equation numerically is a notoriously difficult problem, especially in the high-frequency regime, when the wavelength is much smaller than the size of the domain under study [1]. When solving the problem using a grid-based volume approach like the Finite Element Method (FEM), the number of unknowns becomes so large that the direct solution of the resulting linear system is computationally intractable. On the other hand, most iterative methods that have proved successful for elliptic problems become inefficient when applied to problems with highly oscillatory solutions [1].

Domain Decomposition Methods (DDM) try to combine both direct and iterative approaches, by decomposing the original domain into smaller subdomains over which a direct solution of the Helmholtz problem is possible, and then iterating over the subdomains until the solutions match. When the subdomains do not overlap, the problem can be reformulated in terms of unknown sources on the boundaries between the subdomains, linked to impedance-type transmission conditions between the subdomains. The solution produced by these sources inside the individual subdomains must match the restriction of the solution of the full problem on the subdomains. Such a formulation falls into the framework of Schwarz methods [2–5]; recent improvements in the approximation of the transmission operators have made its convergence rate little sensitive to the wavenumber and discretization density [6,7].

However, a remaining limitation of Schwarz methods is that, even with optimal transmission conditions, the number of iterations increases linearly with the number of subdomains. This limitation has been successfully addressed for certain classes of problems, by adding a component to the algorithm that is known in the DDM community as a “coarse grid” [8–12]. This generic name refers to any technique that enables global sharing of information between subdomains,

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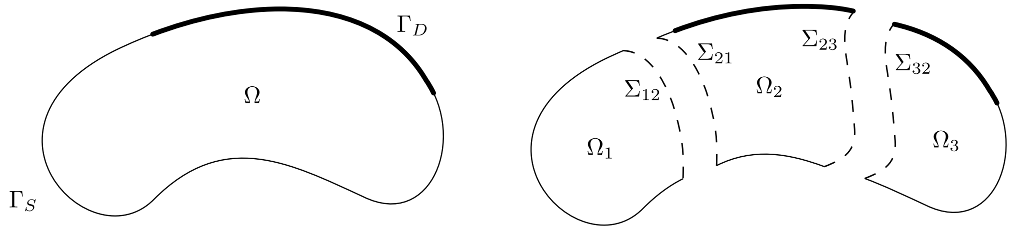

Fig. 1. Left: A general computational domain Ω. Absorbing boundary conditions (ABCs) are used on part of the boundary Γ_S to truncate the domain, and sources can be imposed via Dirichlet conditions on another part Γ_D. Right: Layered decomposition of Ω. Artificial boundaries Σ_{ij} are introduced to separate subdomains Ω_i and Ω_j, such that each domain has 2 neighbors with the exception of the first and last domains.

while the basic additive algorithm only allows local exchange of information, hence hampering convergence (note that multiplicative Schwarz algorithms enable long range exchange of information in one direction only, and cannot guarantee a convergence rate independent of the number of subdomains). While coarse grids have proven to be very effective for Laplace-type problems, designing effective coarse grids for high-frequency Helmholtz problems proves difficult [11,12].

Another way to look at the problem is to recast the DDM as the solution of a linear system, in which case it is natural to search for a preconditioner that would efficiently speed up the convergence of the solver. This paper explores that idea and contributes a way to precondition optimized Schwarz algorithms by taking advantage of recent advances in the development of efficient absorbing boundary conditions for the Helmholtz problem, used as transmission conditions. Starting from a propagation problem for which an exact expression of the optimal transmission condition exists, we will show how such a preconditioner can be designed by approximating the inverse of the iteration operator. It will be interpreted as a double sweep over the subdomains and works by propagating information, just as a coarse grid would. The resulting algorithm is shown to converge very fast, independently of the number of subdomains and frequency. This idea of sweeping to speed up the convergence of iterative Helmholtz solvers, yet not in the context of Schwarz methods, has been proposed in recent works [13–15] where it has shown a comparable effect on the rate of convergence. Our preconditioner has the advantage that, like the original algorithm, it can be implemented as a matrix-free routine and requires no additional preprocessing.

The paper is organized as follows. In Section 2 we begin by formulating the Schwarz algorithm as a linear problem amenable to a solution by Krylov subspace techniques. We then show in Section 3 that the iteration operator that corresponds to the problem formulated in terms of surface unknowns has different properties than the Helmholtz operator, that are especially interesting when a good approximation of the optimal transmission operator (the Dirichlet-to-Neumann map, or DtN map) is available. An iterative solution of this problem can be quickly obtained by means of an efficient preconditioner that exploits these properties, as detailed in Section 4. In Section 5, we summarize the different approximations of the DtN map that will be used as transmission conditions in the algorithm. Section 6 presents numerical results obtained with the proposed method on a variety of test cases.

2. Non-overlapping optimized Schwarz algorithm

The original domain decomposition method introduced by Schwarz [2] makes use of Dirichlet boundary conditions on the artificial interfaces. It is well known that the rate of convergence of this method depends on the size of the overlap between the subdomains and that the method stagnates if the subdomains do not overlap [8]. Convergence without overlap requires Robin or mixed conditions, or more generally impedance-type conditions [16,17], giving rise to so-called non-overlapping optimized Schwarz algorithms [6,5]. Such algorithms benefit from an easy partitioning of the domain and do not require the explicit construction of the normal derivative of the solution, although the treatment of junctions between multiple subdomains (so-called "cross-points") requires special care [18].

In this paper, we take advantage of the structure of particular decompositions, called *layered partitionings*, in which the domain is sliced such that subdomains have at most two neighbors, and the first and last subdomains do not share a boundary (see Fig. 1, right). We first give a detailed description of the basic algorithm, before rewriting it in a form suitable for Krylov acceleration.

2.1. Description of the algorithm

We want to solve the Helmholtz problem with wavenumber k in a domain Ω:

$$\begin{aligned} &-(\Delta + k^2)u = f \quad \text{in } \Omega; \\ &u = u_D \quad \text{on } \Gamma_D; \\ &u \quad \text{is outgoing} \quad \text{on } \Gamma_S. \end{aligned} \tag{1}$$

We consider a layered decomposition of Ω into N non-overlapping slices $\Omega_{i,1\leqslant i\leqslant N}$, with artificial boundaries Σ_{ij} between Ω_i and Ω_j, so that our partitioning contains no loop: $\Omega_i \cap \Omega_j = \emptyset$ if $|i-j| \neq 1$.