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Reconstruction of shapes and impedance functions using few far-field measurements

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ABSTRACT

We consider the reconstruction of complex obstacles from few far-field acoustic measurements. The complex obstacle is characterized by its shape and an impedance function distributed along its boundary through Robin type boundary conditions. This is done by minimizing an objective functional, which is the L^2 distance between the given far-field information g^∞ and the far-field of the scattered wave u^∞ corresponding to the computed shape and impedance function. We design an algorithm to update the shape and the impedance function alternatively along the descent direction of the objective functional. The derivative with respect to the shape or the impedance function involves solving the original Helmholtz problem and the corresponding adjoint problem, where boundary integral methods are used. Further we implement level set methods to update the shape of the obstacle. To combine level set methods and boundary integral methods we perform a parametrization step for a newly updated level set function. In addition since the computed shape derivative is defined only on the boundary of the obstacle, we extend the shape derivative to the whole domain by a linear transport equation. Finally, we demonstrate by numerical experiments that our algorithm reconstruct both shapes and impedance functions quite accurately for non-convex shape obstacles and constant or non-constant impedance functions. The algorithm is also shown to be robust with respect to the initial guess of the shape, the initial guess of the impedance function and even large percentage of noise.

1. Introduction

Let D be a bounded domain of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus \overline{D}$ is connected. We assume that its boundary ∂D is of class C^2. The propagation of time-harmonic acoustic fields in homogeneous cylinder media can be modelled by the Helmholtz equation

$$\Delta u + \kappa^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \tag{1}$$

where $\kappa > 0$ is the wave number. At the obstacle boundary, ∂D, we assume that the total field u satisfies the Robin type boundary condition. That is,

$$\frac{\partial u}{\partial n} + i\kappa\sigma u = 0 \quad \text{on } \partial D \tag{2}$$

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with some impedance function σ where n is the outward unit normal of ∂D. We assume that σ is a real valued C^1-continuous function and has a uniform lower bound $\sigma_- > 0$ on ∂D. The boundary ∂D is referred to be coated.

For a given incident plane wave $u^i(x,d) = e^{i\kappa d\cdot x}$ with incident direction $d \in \mathbb{S}^1$, where $\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$, we look for a solution $u(x,d) := u^i(x,d) + u^s(x,d)$ of (1), (2), where the scattered field u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - i\kappa u^s\right) = 0 \tag{3}$$

with $r = |x|$ and the limit is uniform for all directions $\hat{x} := x/|x| \in \mathbb{S}^1$. It is well known (cf. [7]) that the scattered wave has the asymptotic behavior:

$$u^s(x,d) = \frac{e^{i\kappa r}}{\sqrt{r}} u^\infty(\hat{x},d) + O(r^{-3/2}), \quad r \to \infty, \tag{4}$$

where the function $u^\infty(\hat{x},d)$ is called the far-field of the scattered wave $u^s(x,d)$ corresponding to the incident direction d.

The problem we are considering is formulated as the following inverse scattering problem.

Complex obstacles reconstruction problem. *Given* $u^\infty(\hat{x},d)$ for every $\hat{x} \in S^1$ *and for* K *incident directions* $d = d_1, d_2, \ldots, d_K$,*we want to reconstruct the complex obstacle* $(\partial D, \sigma)$.

In the case where d varies in a connected open subset of S^1, we have uniqueness of the inverse problem, see [19]. This uniqueness issue is largely open if we restrict ourselves to a finite number of incident directions. For some particular situations partial results are known. Indeed, in the case where the Robin boundary condition is replaced by the Dirichlet one (which is "similar" to take σ large in the Robin boundary condition), local uniqueness results for detecting the shapes are obtained, see [8,30,12] as well as local stability results, see [16,17,28]. These results are valid for small obstacles, see [8,16,17], and for close obstacles, see [30,12,28]. In addition, if we know an upper bound of the size of obstacles, then we can estimate the number of incident directions needed to insure uniqueness, see [8,12]. Moreover, if we know in advance that the obstacle D is polygonal, then two incident directions are enough for detecting $(\partial D, \sigma)$, see [21] and the references there, while for the Dirchlet case one incident direction is enough, see [1]. For this particular form of the obstacle, stability estimates are also provided in [25]. For general forms of obstacles and for Robin boundary conditions, the local uniqueness question is still an open issue. In case we know a-priori the obstacle, then a stability result for detecting the surface impedance is given in [29].

The object of this paper is to design a level set type [24] algorithm combined with boundary integral methods to reconstruct $(\partial D, \sigma)$ from few incident directions. Reconstructing shapes by level set methods, introduced by Santosa for inverse problems in [27], has a long history in both shape optimization and inverse scattering fields, see review papers [4,9] for more details. The level set method introduced by Osher and Sethian in [24] tracks the motion of an interface by embedding the interface as the zero level set of a signed distance function. The motion of the interface is matched with the evolution of the zero level set. Therefore, by working with a one dimension higher level set function it is not necessary to track the propagation of the interface, topological changes can occur in a natural manner, and the technique extends easily to three dimensions. However, in our framework of combining level set methods with boundary integral methods we need an explicit boundary representation of the zero level set from the given level set function. Therefore, we do not particularly benefit from level set methods. However, a main justification for the use of the level set method is the possibility of generalization. Note that a parametrization approach has been used in [32], where a Newton method was applied using the derivative with respect to the parametrization basis function. This might be more efficient, but the obvious drawback is the inherent need of a parametrization of the boundary, which requires some sort of a priori guess (such as the number of connected components) of the solution. On the contrary, in our approach the only need for a parametrization of the boundary comes from the computations of the forward problem, more precisely from the use of the boundary integral method for computing the far-field. A parametrization procedure for the inverse problem is not needed when the far-field is computed by finite element methods, for instance. Another possible application of the level set method are near-field problems (e.g. impedance tomography), where finite elements are a usual tool for approximation the forward problem. In these cases we can fully benefit from the advantages of a parametrization free algorithm for the inverse problems, in particularly by allowing topological changes. Therefore, the level set procedure can be applied to multi scattering problems as well. This will be our future work and is illustrated in more detail in Section 4.

Moreover, the novelty of our work lies in that we can reconstruct both the shape D and the impedance function $\sigma(x)$ by using the gradient descent method to minimize a least squares functional related with the given far-field data. To do so we need first to compute derivatives of the minimizing functional with respect to the shape and the impedance function, and then update the level set function via the shape derivative and update the impedance function via the impedance derivative alternatively. This is a non-convex problem and there is no uniqueness guaranteed. Nevertheless our numerical results surprisingly show very good reconstructions of both shapes and impedance functions, for non-convex shapes and non-constant impedance functions.

To find an explicit boundary curve of the zero level set from the given level set function, we have to assume that the obstacle is star-shape like and a point inside the obstacle is known. This is a rather weak assumption and it can be given naturally in some real cases that the location of the obstacle is known. It can also be obtained by other direct and non-iterative imaging methods, such as [15] which uses full far-field data and multiple frequencies to obtain accurate shape reconstructions, or [10] which uses topological derivatives to obtain rough shape reconstructions from full or partial far-field data,