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Monoslope and multislope MUSCL methods for unstructured meshes

Thierry Buffard [a,*], Stéphane Clain [b]

[a] Université Clermont-Ferrand II, Laboratoire de Mathématiques, UMR CNRS 6620, 63177 Aubière cedex, France
[b] Institut de Mathématiques de Toulouse, UMR CNRS 5219, 118 route de Narbonne, 31062 Toulouse cedex, France

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ABSTRACT

We present new MUSCL techniques associated with cell-centered finite volume method on triangular meshes. The first reconstruction consists in calculating one vectorial slope per control volume by a minimization procedure with respect to a prescribed stability condition. The second technique we propose is based on the computation of three scalar slopes per triangle (one per edge) still respecting some stability condition. The resulting algorithm provides a very simple scheme which is extensible to higher dimensional problems. Numerical approximations have been performed to obtain the convergence order for the advection scalar problem whereas we treat a nonlinear vectorial example, namely the Euler system, to show the capacity of the new MUSCL technique to deal with more complex situations.

1. Introduction

Large numerical simulations in industrial framework require efficient but rather simple numerical methods to face the modelling complexity while making easier the implementation. Flexibility is also required to quickly adapt the computation code to new conditions and models. High-resolution methods such as ENO, WENO or Discontinuous Galerkin methods provide very good accuracy. However, the MUSCL technique is more popular in the industrial context due to its natural simplicity and adaptation capacity to respond to modelling evolutions and complexifications.

Monotone Upstream Scheme for Conservation Law technique (MUSCL technique) has been introduced by Van Leer [27] for one-dimensional hyperbolic problems. The main idea is a piecewise linear reconstruction of the solution to achieve higher accurate schemes still preserving the stability: the maximum principle or the Total Variation Diminishing (TVD) property for instance. Initially elaborated for one-dimensional scalar problems, the MUSCL technique combined with a conservative scheme had to preserve the Total Variation of the solution. To this end, slopes are limited to prevent spurious oscillations or overshooting of the numerical approximations [25] and numerous limiters have been proposed [23] in the one-dimensional framework to achieve high-resolution TVD schemes. A first extension of the MUSCL technique to higher dimensions has been proposed using structured meshes where the MUSCL procedure is applied in each direction [8] but the generalization of the Total Variation Diminishing constraint for higher dimensional geometries makes the scheme to be a first-order method [13]. To get around this negative result, a new class of positive schemes have been introduced [24] which ensures a local maximum principle. The concept of Local Extremum Diminishing was then developed by Jameson [15] where he generalizes the notion of incremental scheme with non-negative coefficients for the multi-dimensional situation. For scalar hyperbolic problem, maximum principle naturally derives from the incremental expression and extensions in the Finite Element context have been proposed by Kuzmin and Turek [18].

* Corresponding author.
E-mail addresses: Thierry.Buffard@univ-bpclermont.fr (T. Buffard), clain@mip.ups-tlse.fr (S. Clain).

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An other important point that the reconstruction technique has to address concerns the numerical approximations of hyperbolic system solutions. For the Euler system, density and pressure have to be non-negative to be physically admissible and the shallow-water system requires a non-negative height of water. Numerical approximations have to preserve the density and pressure positivity and specific numerical flux have been designed for this purpose [11]. Extension of the positivity preservation criteria both for second-order finite volume schemes have been also developed [22].

To handle more flexible refinements and allow discretization of complex bounded domains, new MUSCL methods for unstructured meshes have been considered based either on the cell-centered representation [16,9,2] or on the vertex centered representation [5,6]. A linear function is constructed on each element using a gradient prediction which should be limited to prevent oscillations of the numerical solutions [10] (see also [12,19,20] for a mathematical study of the high-order schemes).

The classical MUSCL technique consists of two steps. First, a predicted gradient is computed for each element of the mesh using the neighbouring values. Then the gradient is modified to respect some Maximum Principle or Total Variation Diminishing constraint and provide a vectorial slope on the element. New values are therefore computed on each edge of the element using the linear reconstruction. Finally, an approximation of the flux crossing the interface is performed by employing the two reconstructed values situated on both sides of the edge combined with a monotone numerical flux function. To avoid the predictor–corrector algorithm and obtain some optimal reconstruction, we propose to build the vectorial slope on each element by minimizing a convex functional under stability constraints. The idea is to optimize the slope while respecting the Maximum principle or the Total Variation Diminishing property. We intend in this way to produce the best gradient approximation which respects the stability constraint.

The MUSCL method presented above will be referred to as **monoslope method** since the reconstructed values are obtained using the same vectorial slope on each element. We also introduce a new class of MUSCL method named **multislope method** where we use specific scalar slope for each interface. For a given element, we consider a set of normalized vectors and we use the neighbouring values to compute the scalar slopes representing an approximation of the directional derivatives. The slopes are modified afterwards to respect some stability constraint and finally, the reconstructed values are computed on each edge using the corrected slopes. The main advantage of the method is that we only deal with one-dimensional situations and, as we shall show in the following sections, the scalar slopes are very simple to compute even for higher dimensional geometries.

The remainder of the paper is organized as follows. In Section 2, we introduce the notations we shall use in the sequel to describe the finite volume process on triangular meshes for two-dimensional geometries and we review some classical MUSCL-type methods. In particular, we give a precise description of the Maximum Principle domain and the Total Variation Diminishing domain that we employ to keep the stability condition. Section 3 is devoted to a new monoslope MUSCL method while we describe the multislope MUSCL technique in Section 4. Numerical results are presented for the linear advection problem and the Euler system in Section 5.

2. Second-order monoslope MUSCL method

To illustrate the MUSCL reconstruction, we here introduce the classical advection problem but more complex problems such as nonlinear vectorial systems can of course be considered.

Let $\Omega \subset \mathbb{R}^2$, be a polygonal open bounded set of $\mathbb{R}^2, T > 0$. We denote by $\mathbf{V}(t,x)$ a given $\mathbb{R}^2$ vectorial valued function defined on $Q_T = [0,T] \times \overline{\Omega}$. For $t \in [0,T]$, we set

$$\Gamma^-(t) = \{x \in \partial\Omega; \mathbf{V}(t,x)\cdot\mathbf{n}(x) < 0\}, \quad \Gamma^+(t) = \{x \in \partial\Omega; \mathbf{V}(t,x)\cdot\mathbf{n}(x) \geqslant 0\},$$

with $x = (x_1, x_2)$ a generic point of Ω and $\mathbf{n}$ the outwards normal on the boundary $\partial\Omega$.

We consider the advection problem: find $U(t,x)$ a real valued function defined on Q_T such that

$$\begin{aligned}
&\partial_t U + \nabla\cdot(\mathbf{V}U) = 0 \quad \text{in }]0,T[\times\Omega,\\
&U(t=0,\cdot) = U_0(\cdot) \quad \text{in } \Omega,\\
&U(t,\cdot) = U_b(t,\cdot) \quad \text{in } \Gamma^-(t),\ t \in]0,T],
\end{aligned}$$

where U_0 and U_b are given functions.

To deal with the numerical approximation, we introduce the following ingredients (see Fig. 1). $\mathcal{T}_h$ is a discretization of Ω with triangles K_i of centroid $\mathbf{B}_i, i = 1,\dots,N$ where N is the number of mesh elements. For a given i, $\nu(i)$ represents the index set of the common edge elements $K_j \in \mathcal{T}_h, j \in \nu(i)$ where $S_{ij} = \overline{K}_j \cap \overline{K}_i$ stands for the common edge with midpoint $\mathbf{M}_{ij}$.

We assume furthermore that the mesh satisfies the following hypothesis (see Fig. 2):

$$(\mathcal{H}) \quad \begin{cases} \text{For any } K_i \in \mathcal{T}_h \text{ such that } |\nu(i)| = 3, \text{ point } \mathbf{B}_i \text{ is strictly} \\ \text{inside the convex set defined by the points } \mathbf{B}_j, j \in \nu(i). \end{cases}$$

Remark 1. Hypothesis $(\mathcal{H})$ yields that any two of the three vectors $B_iB_j, j \in \nu(i)$ defines a basis of $\mathbb{R}^2$. Such a property is essential to define the monoslope MUSCL method and it is less restrictive than Hypothesis $(\mathcal{H})$. Nevertheless, the multislope