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Positivity-preserving high order finite difference WENO schemes for compressible Euler equations ☆

Xiangxiong Zhang [1], Chi-Wang Shu *

Division of Applied Mathematics, Brown University, Providence, RI 02912, United States

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ABSTRACT

In Zhang and Shu (2010) [20], Zhang and Shu (2011) [21] and Zhang et al. (in press) [23], we constructed uniformly high order accurate discontinuous Galerkin (DG) and finite volume schemes which preserve positivity of density and pressure for the Euler equations of compressible gas dynamics. In this paper, we present an extension of this framework to construct positivity-preserving high order essentially non-oscillatory (ENO) and weighted essentially non-oscillatory (WENO) finite difference schemes for compressible Euler equations. General equations of state and source terms are also discussed. Numerical tests of the fifth order finite difference WENO scheme are reported to demonstrate the good behavior of such schemes.

1. Introduction

In this paper we are interested in the Euler equations, the one dimensional version for the perfect gas being given by

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x = 0, \quad t \geqslant 0, \quad x \in \mathbb{R}, \tag{1.1}$$

$$\mathbf{w} = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix}, \quad \mathbf{f}(\mathbf{w}) = \begin{pmatrix} m \\ \rho u^2 + p \\ (E+p)u \end{pmatrix} \tag{1.2}$$

with

$$m = \rho u, \quad E = \frac{1}{2}\rho u^2 + \rho e, \quad p = (\gamma - 1)\rho e,$$

where ρ is the density, u is the velocity, m is the momentum, E is the total energy, p is the pressure, e is the internal energy, and $\gamma > 1$ is a constant ($\gamma = 1.4$ for the air). The speed of sound is given by $c = \sqrt{\gamma p/\rho}$ and the three eigenvalues of the Jacobian $\mathbf{f}'(\mathbf{w})$ are $u - c$, u and $u + c$.

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* Corresponding author. Tel.: +1 401 863 2549; fax: +1 401 863 1355.
E-mail addresses: zhangxx@dam.brown.edu (X. Zhang), shu@dam.brown.edu (C.-W. Shu).
[1] Current Address: Department of Mathematics, MIT, Cambridge, MA 02139, United States.

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In a conservative numerical scheme, the internal energy is obtained by subtracting the kinetic energy from the total energy, thus the resulting pressure may be negative, for example, for problems in which the dominant energy is kinetic. Negative density may often emerge in computing blast waves. Physically, the density ρ and the pressure p should both be positive. The eigenvalues of the Jacobian will become imaginary if density or pressure is negative so the initial value problem for the linearized system will be ill-posed. This explains why failure of preserving positivity of density or pressure may cause blow-ups of the numerical algorithm.

Replacing the negative density or negative pressure by positive ones is neither a conservative cure nor a stable solution. Therefore, it is highly important to design a conservative positivity-preserving scheme. First order and second order positivity-preserving schemes were well studied, e.g. [4,11]. A general framework for constructing arbitrarily high order positivity-preserving discontinuous Galerkin (DG) and finite volume schemes was proposed recently in [20]. This framework can be easily generalized, for instance, to unstructured meshes [23], and to general equations of state and Euler system with source terms [21].

Generalization of the positivity-preserving method in [20] to high order finite difference schemes is not straightforward. However, in some applications where high order schemes are preferred, for example, cosmological simulation [5], finite difference WENO schemes [10] is more favored than DG schemes [2,3] and the finite volume WENO scheme [12,15] due to their smaller memory cost (compared to DG) and smaller computational cost (compared both to finite volume schemes and to DG schemes) for multi-dimensional problems, see for example a comparison in [1] in the context of ENO schemes.

In this paper, we will follow the idea in [20] to construct positivity-preserving high order finite difference WENO schemes. We will show that by adopting the same simple limiter as in [20], to a slightly different version of finite difference WENO schemes from the one in [10], the final scheme will keep the positivity of density and pressure without losing conservation. A conservative positivity-preserving scheme is L^1-stable, see [22]. The limiter will not destroy the high order accuracy of the WENO scheme for smooth solutions without vacuum. All the results also hold for finite difference ENO schemes [16].

The paper is organized as follows. In Section 2, we briefly review the positivity-preserving finite volume schemes in [20] and the finite difference WENO scheme in [10]. Then we introduce positivity-preserving finite difference schemes in one space dimension for the perfect gas in Section 3. In Section 4, we discuss a straightforward extension to multi-dimensions, general equations of state and source terms. In Section 5, numerical tests of the fifth order WENO schemes for some very demanding problems are shown. Concluding remarks are given in Section 6.

2. Preliminaries

2.1. Review of positivity-preserving high order finite volume WENO schemes

We first briefly review the basic idea in [20,22] for finite volume WENO schemes. Consider the Euler Eq. (1.1) in more detail. Let $p(\mathbf{w}) = (\gamma-1)\left(E - \frac{1}{2}\frac{m^2}{\rho}\right)$ be the pressure function. It can be easily verified that p is a concave function of $\mathbf{w} = (\rho, m, E)^T$ if $\rho > 0$. For $\mathbf{w}_1 = (\rho_1, m_1, E_1)^T$ and $\mathbf{w}_2 = (\rho_2, m_2, E_2)^T$, Jensen's inequality implies, for $0 \leqslant s \leqslant 1$,

$$p(s\mathbf{w}_1 + (1-s)\mathbf{w}_2) \geqslant sp(\mathbf{w}_1) + (1-s)p(\mathbf{w}_2), \quad \text{if} \quad \rho_1 > 0, \quad \rho_2 > 0. \tag{2.1}$$

Define the set of admissible states by

$$G = \left\{ \mathbf{w} = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix} \middle| \rho > 0 \quad \text{and } p = (\gamma-1)\left(E - \frac{1}{2}\frac{m^2}{\rho}\right) > 0 \right\},$$

then G is a convex set. We want to construct finite volume WENO schemes producing solutions in the set G. Notice that the condition $p > 0$ in the definition of set G can be changed to $p \geqslant 0$ without affecting convexity.

The time discretization is chosen as the high order strong stability preserving (SSP) methods [14,16,8,9] which are convex combinations of Euler forward. Thus we only need to discuss the Euler forward since G is convex.

A general high order finite volume scheme has the following form

$$\bar{\mathbf{w}}_i^{n+1} = \bar{\mathbf{w}}_i^n - \lambda\left[\hat{\mathbf{f}}\left(\mathbf{w}^-_{i+\frac{1}{2}}, \mathbf{w}^+_{i+\frac{1}{2}}\right) - \hat{\mathbf{f}}\left(\mathbf{w}^-_{i-\frac{1}{2}}, \mathbf{w}^+_{i-\frac{1}{2}}\right)\right], \tag{2.2}$$

where $\hat{\mathbf{f}}$ is a positivity preserving flux, for instance, Lax–Friedrichs flux, $\bar{\mathbf{w}}_i^n$ is the approximation to the cell average of the exact solution $\mathbf{v}(x,t)$ in the cell $I_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ at time level n, and $\mathbf{w}^-_{i+\frac{1}{2}}, \mathbf{w}^+_{i+\frac{1}{2}}$ are the high order approximations of the point values $\mathbf{v}\left(x_{i+\frac{1}{2}}, t^n\right)$ within the cells I_i and I_{i+1} respectively. These values are reconstructed from the cell averages $\bar{\mathbf{w}}_i^n$ by the WENO reconstruction. We assume that there is a polynomial vector $\mathbf{q}_i(x) = (\rho_i(x), m_i(x), E_i(x))^T$ with degree k which are $(k+1)$-th order accurate approximations to smooth exact solutions $\mathbf{v}(x,t)$ on I_i, and satisfies that $\bar{\mathbf{w}}_i^n$ is the cell average of $\mathbf{q}_i(x)$ on I_i, $\mathbf{w}^+_{i-\frac{1}{2}} = \mathbf{q}_i\left(x_{i-\frac{1}{2}}\right)$ and $\mathbf{w}^-_{i+\frac{1}{2}} = \mathbf{q}_i\left(x_{i+\frac{1}{2}}\right)$. The existence of such polynomials can be established by interpolation for WENO schemes. For example, for the fifth order WENO scheme, there is a unique vector of polynomials of degree four $\mathbf{q}_i(x)$ satisfying $\mathbf{q}_i\left(x_{i-\frac{1}{2}}\right) = \mathbf{w}^+_{i-\frac{1}{2}}, \mathbf{q}_i\left(x_{i+\frac{1}{2}}\right) = \mathbf{w}^-_{i+\frac{1}{2}}$ and