Journal of Computational Physics 229 (2010) 978–1016

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Analysis of Godunov type schemes applied to the compressible Euler system at low Mach number

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ARTICLE INFO

Article history:
Received 17 November 2008
Received in revised form 25 September 2009
Accepted 29 September 2009
Available online 9 October 2009

Keywords:
Compressible Euler system
Low Mach number flow
Roe scheme
Roe–Turkel scheme
AUSM+-up scheme
Linear wave equation
Hodge decomposition

ABSTRACT

We propose a theoretical framework to clearly explain the inaccuracy of Godunov type schemes applied to the compressible Euler system at low Mach number on a Cartesian mesh. In particular, we clearly explain why this inaccuracy problem concerns the 2D or 3D geometry and does not concern the 1D geometry. The theoretical arguments are based on the Hodge decomposition, on the fact that an appropriate well-prepared subspace is invariant for the linear wave equation and on the notion of first-order modified equation. This theoretical approach allows to propose a simple modification that can be applied to any colocated scheme of Godunov type or not in order to define a large class of colocated schemes accurate at low Mach number on any mesh. It also allows to justify colocated schemes that are accurate at low Mach number as, for example, the Roe–Turkel and the AUSM+-up schemes, and to find a link with a colocated incompressible scheme stabilized with a Brezzi–Pitkäranta type stabilization. Numerical results justify the theoretical arguments proposed in this paper.

1. Introduction

It is sometimes essential to take into account compressibility phenomena even when a flow is at low Mach number. Moreover, a flow may also be at low Mach number only in a part of the physical domain. In such situations, it is important to model the flow with the compressible Navier–Stokes system.

Since the compressible Euler system is the compressible Navier–Stokes system without the physical diffusive terms, any study at low Mach number may firstly concern the compressible Euler system. Conservative finite volume schemes as Godunov type schemes [1–3] are colocated schemes that are well adapted to capture shock waves solution of the compressible Euler system. Nevertheless, it is well known that Godunov type schemes are not accurate at low Mach number [4–10]. This inaccuracy is characterized by the creation of spurious pressure and velocity waves that avoid the velocity field to be close to a divergence-free field. Let us note that this inaccuracy concerns other colocated compressible schemes [11,12]. This inaccuracy problem comes from a loss of information between the continuous and discrete levels. Indeed, at the continuous level, the solution of the compressible Euler system converges toward the solution of the incompressible Euler system when the Mach number goes to zero. Nevertheless, many numerical experiments show that at low Mach number, the numerical solution given by a colocated compressible scheme may be far from an incompressible numerical solution. In [6,10], the origin of the inaccuracy is explained by studying the 1D-formulation of Godunov type schemes through a formal asymptotic development based on the Mach number. Such studies allow, firstly, to partly find the origin of the inaccuracy and, secondly, to

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doi:10.1016/j.jcp.2009.09.044

propose modifications of Godunov type schemes to increase the accuracy at low Mach number. For example, in [10], the Roe scheme with pressure correction is proposed. This scheme consists in modifying the numerical viscosity by adding a pressure correction on the momentum equation. The Roe–Turkel [4,5,7] and VFRoe–Turkel [8,9] schemes are other modified Godunov type schemes; the schemes proposed in [6,13] are similar to the Roe–Turkel scheme. All these schemes are built by modifying the numerical diffusion with the Turkel preconditioning matrix [14,15]. In [7–9], the Roe–Turkel and VFRoe–Turkel schemes are justified with a formal asymptotic development based on the Mach number; *idem* in [6]. Let us underline that the Roe–Turkel and VFRoe–Turkel schemes are much more complicated to implement than the Roe scheme with pressure correction proposed in [10]. All these modified Godunov type schemes give good numerical results. In [12], Liou proposes a flux splitting type scheme – named AUSM$^+$-up scheme – that is also accurate at low Mach number. In [16–18], other colocated schemes accurate at low Mach number are proposed.

We propose in this work a theoretical framework to clearly explain the inaccuracy of Godunov type schemes at low Mach number when the mesh is Cartesian. This theoretical framework is based on a Schochet's result [19], on the Hodge decomposition, on the fact that an appropriate well-prepared subspace is invariant for the linear wave equation and on the notion of first-order modified equation. In particular, we show that the inaccuracy problem does not exist in 1D geometry that is to say only exists in 2D or 3D geometry. Let us note that in [8], the linear wave equation is also studied by Guillard et al. But, our analysis seems to be more direct and, especially, allows to clearly identify the invariance property that is the key argument. Our analysis underlines also that the inaccuracy of Godunov type schemes at low Mach number can be explained (at least partly) with simple linear arguments by only analyzing the linear wave equation. Nevertheless, it is important to note that the proposed theoretical approach would not have been possible without the previous works of Schochet and Guillard et al. We also propose a formal approach that is coherent with our theoretical results and we compare this formal approach to other formal approaches [4,6,8,10] that seem to be less precise. The proposed theoretical approach allows us to conjecture the existence of a large class of colocated schemes that are accurate at low Mach number on any mesh. This class is named *low Mach X schemes* and consists in a simple modification of any *X* scheme of Godunov type ($X \in \{Roe, VFRoe, \ldots\}$) or not ($X$ = kinetic scheme [20] for example). Numerical results show that the *low Mach Roe scheme* and the *low Mach VFRoe scheme* (i.e. $X = Roe$ and $X = VFRoe$) are accurate at low Mach number. Let us underline that the proposed theoretical approach justifies the Roe–Turkel [4,5,7] (or VFRoe–Turkel [8,9]) scheme and the Roe scheme with pressure correction [10] since, in the linear case, they are respectively similar and identical to the *low Mach Godunov scheme*. Moreover, it is also possible to prove that the schemes proposed in [12,16–18] are similar or identical to a *low Mach X scheme* when the Mach number goes to zero.

The outline of this paper is the following. In Section 2, we recall the derivation of the incompressible Euler system from the compressible Euler system with formal arguments and with theoretical arguments due to Schochet [19]. Then, we show how to prove the Schochet's result in the linear case with simple arguments. This Section 2 allows us to introduce the notion of well-prepared subspace and a sufficient condition allowing to avoid the creation of spurious pressure and velocity waves. In Section 3, we propose a simple formal approach showing that the inaccuracy problem only exists in 2D or 3D geometry. In Section 4, we explain with theoretical arguments why Godunov type schemes are inaccurate only in 2D or 3D geometry. In Section 5, we propose to apply a simple modification to any colocated *X* scheme in order to define a large class of *low Mach X schemes* accurate at low Mach number. This modification is justified in the case of the linear Godunov scheme. Then, we show that Roe–Turkel type schemes [7–9] and Godunov type schemes with pressure correction [10] are respectively similar and identical to the *low Mach Godunov scheme* in the linear case. We also underline that the AUSM$^+$-up scheme [12] and the colocated schemes proposed in [16–18] are respectively similar and identical to a *low Mach X scheme* when the Mach number goes to zero. Moreover, we formally show that there exists a link between the proposed *low Mach Godunov scheme* and a colocated incompressible scheme stabilized with a Brezzi–Pitkäranta type stabilization [21]. In Section 6, we introduce a more general theoretical framework allowing to propose other colocated schemes that do not create spurious pressure and velocity waves. Finally, we show in Section 7 numerical results that justify the theoretical arguments proposed in this paper.

2. Convergence of a compressible flow toward an incompressible flow

We recall the classical formal derivation of the incompressible Euler system from the compressible Euler system [22]. This recall will allow us to formally introduce the notion of well-prepared initial condition. This will be essential in the sequel. Then, we recall the theoretical derivation proposed in [19] and we prove the result in the linear case with simple arguments. This linear study will allow us to, firstly, define in Section 2.4 a sufficient condition to avoid the creation of spurious (pressure and velocity[1]) waves, to, secondly, explain in Section 4 the inaccuracy of Godunov type schemes at low Mach number and to, thirdly, propose in Section 5 a simple curative solution. In Section 6, we will propose a weaker sufficient condition to avoid the creation of spurious waves.

2.1. Formal derivation

The dimensionless compressible Euler system is given by

[1] In the sequel, we write *spurious waves* for *spurious pressure and velocity waves*.