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The monotonic Quartic Spline Method (QSM) for conservative transport problems ☆

M. Zerroukat *, A. Staniforth, N. Wood

Met Office, FitzRoy Road, Exeter EX1 3PB, United Kingdom

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ABSTRACT

A quartic spline based remapping algorithm is developed and illustrative tests of it are presented herein. To ensure mass conservation, the scheme solves an integral form of the transport equation rather than the differential form. The integrals are computed from reconstructed quartic splines with mass conservation constraints. For higher dimensions, this remapping can be used within a standard directional splitting methodology or within the flow-dependent cascade splitting approach. A high-order grid and sub-grid based monotonic filter is also incorporated into the overall scheme. This filter is independent of the underlying spline representation adopted here, and is of more general application.

1. Introduction

Remapping algorithms, such as the widely used Piecewise Parabolic Method (PPM) [1], are an important component in many advection schemes for conservative transport. These remappings are also the building blocks of many of the inherently conserving semi-Lagrangian schemes [2–11].

An alternative to PPM, based on the Parabolic Spline Method (PSM), was presented in [12] and demonstrated in [13] for two-dimensional conservative transport in Cartesian and spherical geometries. PSM is similar to PPM, but more accurate (due to its "best approximation" property), whilst being 60% more efficient [12]. PSM also incorporates a more selective, and less damping, monotonic filter than that used in the original PPM [1]. PSM achieves monotonicity without (except in extreme cases) reducing the order of the piecewise polynomial, and it well captures steep gradients and curvature without recourse to artificial steepening.

The goal of the present paper is to generalise the PSM remapping algorithm to higher order, for increased accuracy. Being based on a quartic spline, the resulting algorithm is termed the Quartic Spline Method (QSM). Similarly to PSM, QSM also has

* Corresponding author.
E-mail address: Mohamed.Zerroukat@metoffice.gov.uk (M. Zerroukat).
URL: http://www.metoffice.gov.uk (M. Zerroukat).

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a "best approximation" property: whereas PSM is optimal within the class of *second*-order polynomial representations of density, QSM is optimal within the class of *fourth*-order ones.

The rest of the paper is organised as follows: Section 2 details the QSM remapping algorithm and its properties; its monotonic filter is described in Section 3; results using the proposed scheme are presented in Section 4 and compared with those using PSM; and conclusions are summarised in Section 5.

2. The Quartic Spline Method (QSM)

2.1. Problem definition

Consider passive 1D conservative transport of a scalar quantity ρ governed, in the absence of sources and sinks, by

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(u\rho)=0, \tag{1}$$

where $\rho(x,t)$ is the density (amount of scalar per unit length) of the transported quantity, and $u(x,t)$ is the transporting velocity field. Assume a finite fluid volume bounded by two arbitrary boundaries $x_1=x_1(x,t)$ and $x_2=x_2(x,t)$ moving with the fluid, so that

$$\frac{dx_1}{dt}=u(x_1,t), \quad \frac{dx_2}{dt}=u(x_2,t). \tag{2}$$

Integrating (1) with respect to x between two arbitrary moving boundaries $x_L=x_L(x,t)$ and $x_R=x_R(x,t)$, and making use of Leibniz' rule, then leads [5] to the classical integral form of the tracer conservation equation

$$\frac{d\mathfrak{M}(x_L,x_R,t)}{dt}\equiv\frac{d}{dt}\left(\int_{x_L(t)}^{x_R(t)}\rho(x,t)dx\right)=0. \tag{3}$$

Eq. (3) simply states that the mass $\mathfrak{M}(x_L,x_R,t)$ contained between any two boundaries, $x_L(t)$ and $x_R(t)$, that move with the fluid, is invariant in time, i.e. $\mathfrak{M}$ is conserved.

Since $x_L(t)$ and $x_R(t)$ in (3) are any two points travelling with the fluid, one can consider that these moving boundaries instantaneously coincide at time t^{n+1} with the boundaries of an Eulerian Control Volume (ECV). Their upstream positions $x_L(t^n)$ and $x_R(t^n)$ at time t^n then form the left and right boundaries of the associated upstream Lagrangian Control Volume (LCV). In other words, since the fluid is a continuum, then the fluid contained in the Lagrangian segment $\left[x_L^d,x_R^d\right]\equiv[x_L(t^n),x_R(t^n)]$ is completely transported to the Eulerian segment $[x_L(t^{n+1}),x_R(t^{n+1})]$ (this provides the basis of the SLICE scheme [2]).

To discretise (3), consider the general case where the computational 1D domain $\Omega=[x_{min},x_{max}]$ is subdivided into N ECV's with (possibly unequal) spacing $h_i\equiv x_{i+1/2}-x_{i-1/2}$ $(i=1,2,\ldots,N)$, where $x_{i-1/2}$ and $x_{i+1/2}$ are respectively the left and right boundaries of ECV_i. For a closed domain, the left boundary is at $x=x_{1/2}$ and the right boundary at $x=x_{N+1/2}$. For a periodic domain, $x_{N+i+1/2}\equiv x_{i-1/2}$ $(i=0,\pm1,\pm2,\ldots)$.

Defining the gridbox-averaged density at time t as

$$\bar{\rho}_i(t)\equiv\frac{1}{h_i}\int_{x_{i-1/2}}^{x_{i+1/2}}\rho(x,t)dx\equiv\frac{1}{h_i}\mathfrak{M}(x_{i-1/2},x_{i+1/2},t)\equiv\frac{1}{h_i}\mathfrak{M}_i, \tag{4}$$

the time-discretisation of (3) can then be rewritten as

$$\bar{\rho}_i^{n+1}\equiv\bar{\rho}_i(t^{n+1})\equiv\frac{1}{h_i}(\mathfrak{M}_i)^{n+1}=\frac{1}{h_i}\left(\mathfrak{M}_i^d\right)^n, \tag{5}$$

where

$$\mathfrak{M}_i^d\equiv\int_{x_{i-1/2}^d}^{x_{i+1/2}^d}\rho(x,t)dx. \tag{6}$$

Here superscript n denotes evaluation at time t^n, superscript d denotes association with a departure-point value (as in semi-Lagrangian schemes [14]), and $x_{i-1/2}^d$ and $x_{i+1/2}^d$ are respectively the left- and right-hand boundaries of LCV_i at time t^n, determined from numerical integration of (2) – see e.g. [14].

In general, the shape of $\rho(x,t^n)$ is not known *a priori*, and hence (6) cannot be evaluated. Instead piecewise polynomials that use the given discrete gridbox-averaged values can be reconstructed. Previous approaches have used either piecewise constant, piecewise linear [15], piecewise parabolic [1,12] or piecewise cubic [2,8] polynomials. Herein a Quartic Spline Method is proposed.