Journal of Computational Physics 227 (2008) 8736–8752

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Time-reversibility of the Euler equations as a benchmark for energy conserving schemes

M. Duponcheel [a,*], P. Orlandi [b], G. Winckelmans [a]

[a] Université catholique de Louvain (UCL), Mechanical Engineering Department, Division TERM, Center for Systems Engineering and Applied Mechanics (CESAME), 1348 Louvain-la-Neuve, Belgium
[b] Università di Roma "La Sapienza", Department of Mechanics and Aeronautics, 00184 Roma, Italy

ARTICLE INFO

Article history:
Received 11 May 2007
Received in revised form 25 February 2008
Accepted 20 June 2008
Available online 28 June 2008

Keywords:
Euler equations
Time reversibility
Taylor–Green vortex
Energy conservation
Navier–Stokes solvers
Discretization

ABSTRACT

The three-dimensional incompressible Euler equations are time-reversible. This property should be preserved as well as possible by numerical discretizations. This article investigates the time-reversibility properties of various solvers designed for incompressible Navier–Stokes computations. The test case is the inviscid Taylor–Green vortex, which becomes "turbulent" before the time is reversed to try to recover the initial condition. The simulations are performed using high and low order finite difference solvers as well as using a pseudo-spectral solver. Various time-stepping schemes are also investigated. Although the flow statistics are significantly affected by the accuracy of the space discretization, the time-reversibility is not because most space-discretizations are time-reversible for an exact time-stepping. The crucial factor for time-reversibility is the accuracy of the time-stepping scheme and its interaction with the space-discretization. Furthermore, an important practical requirement for the solver is to be energy conserving in order to avoid numerical instability. An energy conserving solver using an accurate time-stepping is then able to go back almost perfectly from a complex "turbulent" flow to the simple initial condition. Therefore, we propose that this constitutes a severe and useful benchmark that Navier–Stokes solvers should challenge. The present investigations and their conclusions are also supported by parallel 1-D investigations, using the non-linear convection equation (inviscid Burgers) and the linear convection equation.

1. Introduction

It is easily verified that the incompressible continuous Euler equations

$$\nabla \cdot \mathbf{u} = 0, \tag{1a}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla P, \tag{1b}$$

with $P = p/\rho$ the reduced pressure, are time-reversible. If $(\mathbf{u}(\mathbf{x},t), p(\mathbf{x},t))$ is a solution of the system, then $(-\mathbf{u}(\mathbf{x},-t), p(\mathbf{x},-t))$ is also a solution. Therefore, if $\mathbf{u}^*(\mathbf{x})$ is the solution at time t^* of the problem with an initial condition $\mathbf{u}_0(\mathbf{x})$ then $-\mathbf{u}_0(\mathbf{x})$ is the solution at time t^* of the problem with an initial condition $-\mathbf{u}^*(\mathbf{x})$. It is legitimate to expect a numerical discretization of the Euler equations to preserve this time-reversibility property. Note also that the time-reversibility property was previously used by Carati et al. [1] to assess models for explicitly filtered LES because the explicit filtering does not alter the time-reversibility of the equations.

* Corresponding author. Tel.: +32 10 474031; fax: +32 10 452692.
E-mail address: matthieu.duponcheel@uclouvain.be (M. Duponcheel).

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2008.06.020

The aim of this paper is to investigate the time-reversibility property of various Navier–Stokes solvers running with the viscosity deliberately set to zero and without any subgrid-scale model. A fully periodic test case was chosen to avoid the possible influence of the boundary conditions. The present study is thus entirely focused on the behavior of the discretization of the convective term and its interaction with the time stepping scheme. This is indeed the more relevant term in high Reynolds number flows.

The initial condition of the present investigation is the simple analytical Taylor–Green vortex. Then, the flow freely evolves and eventually becomes "turbulent". Small-scales are generated and, since there is no viscosity, the energy spectrum $E(k)$ tends to a k^2 behavior at the high wave numbers. Then, the sign of the velocities is changed and the simulation is run further. This is equivalent to going back in time. The ability of the solver to recover the initial condition is then assessed.

Three different energy-conserving space discretizations are compared in the present study: two centered finite difference schemes and a pseudo-spectral method. The first finite difference scheme is second order accurate whereas the second one is fourth order accurate. In all three solvers, two time-stepping schemes are available: a second order Adams–Bashforth scheme (AB2) and a third order Runge–Kutta method (RK3).

It is observed that the accuracy of the spatial discretization does not directly influence the time-reversibility of the solver; even though the flow statistics strongly depend on it. An energy conserving scheme is however absolutely required since, on the one hand, any spurious injection of energy leads to a fast blow up of the computation and, on the other hand, a systematic dissipation of energy obviously makes it impossible to recover the initial condition. Provided the energy is conserved, the time-reversibility mainly depends on the accuracy of the time discretization scheme because it is shown that the solvers would be perfectly reversible if the time-stepping was exact. Hence, for the same spatial discretization, RK3 performs much better than AB2 (which is known to be slightly unstable for purely convective linear problems, even when using a small time step, as done is this investigation). However, the space discretization also interferes with the time-stepping because the accuracy of the time-stepping depends on its eigenvalues. Consequently, in all cases, the present second order finite difference solver is the best at recovering the initial condition among the studied schemes. As the lack of exact energy conservation of the discrete solver is also due to time stepping errors, there is a strong correlation between energy conservation and time-reversibility.

To recover well enough the initial condition, one needs an energy conserving space discretization and an accurate time-stepping. This also ensures that the energy will be very well conserved. Therefore, the present time-reversibility test is proposed as a sensitive test for energy conserving schemes. Those are especially important for Navier–Stokes solvers and also for LES solvers that use explicit subgrid-scale modeling. They are also important to study the non-linear dynamics of the Euler equation (e.g., see [2]) and to try to give answers to fundamental questions such as the possible finite time singularity in the Euler equations. Here, especially, the effects of the numerics must be sufficiently understood before any definite answer can be given.

This paper is organized as follows. Section 2 briefly describes the solvers assessed in this study. Then, Section 3 presents the Taylor–Green vortex test case and the numerical parameters used. The results obtained on this test case are presented in Section 4. They are further analyzed in Section 5 where they are compared to 1D results in order to clearly identify the factors governing the time-reversibility properties.

2. Numerical methods

This section describes first the two finite difference solvers (referred to as FD2 and FD4) and then the pseudo-spectral solver (PS) used in this investigation.

2.1. Finite difference solvers

Both finite difference codes solve the incompressible Navier–Stokes equations on Cartesian MAC grids. The equations are integrated in time using a fractional-step method introducing the pressure gradient in the computation of the intermediate velocity. It was called the "delta" form for the pressure by Lee et al. [3]. This form allows simple boundary conditions for the pressure and the intermediate velocity field. When the convective term is integrated using AB2, the time-stepping scheme reads

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = -\frac{1}{2}\left(3\mathbf{H}^n - \mathbf{H}^{n-1}\right) - \nabla P^n, \tag{2a}$$

$$\nabla^2\varphi = \frac{1}{\Delta t}\nabla\cdot\mathbf{u}^*, \tag{2b}$$

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = -\nabla\varphi, \tag{2c}$$

$$P^{n+1} = P^n + \varphi, \tag{2d}$$

where $\mathbf{H}^n$ is the convective term, $\mathbf{u}^*$ is the intermediate velocity field and P^n is the reduced pressure. Two time-stepping schemes are available in both solvers: AB2 and RK3. In RK3, the divergence-free constraint in enforced at each substep.