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Stable and accurate wave-propagation in discontinuous media

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ABSTRACT

A time stable discretization is derived for the second-order wave equation with discontinuous coefficients. The discontinuity corresponds to inhomogeneity in the underlying medium and is treated by splitting the domain. Each (homogeneous) sub domain is discretized using narrow-diagonal summation by parts operators and, then, patched to its neighbors by using a penalty method, leading to fully explicit time integration. This discretization yields a time stable and efficient scheme. The analysis is verified by numerical simulations in one-dimension using high-order finite difference discretizations, and in three-dimensions using an unstructured finite volume discretization.

1. Introduction

In many applications, such as general relativity [33,3], seismology [15,36], oceanography [27], acoustics [35,30,7,1,8] and electromagnetics [37,9], the underlying equations are systems of second-order hyperbolic partial differential equations. However, as pointed out in [18], with very few exceptions the equations are rewritten and solved as a system of first-order equations. There are three obvious drawbacks with this approach: (1) the number of unknowns is doubled, (2) spurious oscillations due to unresolved features might be introduced, and (3) double resolution (both in time and in each of the spatial dimensions) is required to obtain the same accuracy. The reasons for solving the equations on first-order form are most likely related to the maturity of CFD, that has evolved during the last 40 years. Many of the stability issues for first-order hyperbolic problems have already been addressed.

For wave-propagation problems, the computational domain is often large compared to the wavelengths, which means that waves have to travel long distances (or correspondingly long times). It can be shown that high-order accurate time marching methods, as well as high-order spatially accurate schemes (at least third-order) are more efficient [21] for problems on smooth domains. Such schemes, although they might be G–K–S stable [10] (convergence to the true solution as $\Delta x \to 0$), may exhibit a non-physical growth in time [4], for realistic mesh sizes. It is therefore important to devise schemes that do not allow a growth in time that is not called for by the differential equation. Such schemes are called strictly (or time) stable.

High-order accurate finite difference methods (HOFDM) are widely used for hyperbolic problems written on first-order form. For problems with discontinuous coefficients, the formal order of accuracy reduces to first-order [11,12,2] with no special treatment of the discontinuity. In this paper we will focus the attention to second-order formulations of the acoustic

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wave equation in discontinuous media. One of the main motivations with this paper is to present a method that will recover high-order accuracy in the presence of discontinuous coefficients.

Traditionally, there have been essentially two approaches of handling the discontinuity, sometimes referred to as the heterogeneous and the homogeneous formulations [15]. In the heterogeneous approach [35,30,7], the discontinuity (here denoted *discontinuous interface*) is treated by taking an average "smoothing" of the spatially varying coefficients to recover stability. The benefit with this technique is that irregular shaped discontinuous interfaces are handled with no special treatment. However, the formal order of accuracy reduces to first-order.

The second approach to handle the discontinuity is to employ a domain decomposition technique and solve for the interface (jump) conditions. There are different techniques of imposing the interface conditions. In [1] a second-order FD method is introduced where the solution is based on the introduction of auxiliary Lagrange multipliers. A drawback with this technique is that a huge system of linear equations has to be solved at each time-step. It is unclear if this technique can be extended to handle irregular shaped discontinuous interfaces, and how to obtain higher-order accuracy. A strictly stable HODFM for the wave equations in discontinuous media was constructed in [24] by combining second-derivative summation-by-parts (SBP) operators (constructed in [23]) with the projection method [28,29] to impose the boundary and the discontinuous interface (jump) conditions. The drawback with this approach is that it cannot easily be extended to handle variable coefficients (except pice-wise constant coefficients), complex geometries and irregular shaped discontinuous interfaces. In [18,19,17,16] a second-order accurate FD method for the acoustic wave equation on second-order form is constructed, where the discontinuity and complex geometry are handled by embedding the domain into a Cartesian grid, making use of ghost-points and Lagrange interpolation to impose the boundary and interface conditions. It is unclear if the embedded boundary method can be extended to higher-order accuracy. Another good candidate is the discontinuous Galerkin (DG) method, which combines both unstructured capability and higher-order accuracy (also in discontinuous media). DG have been implemented successfully in 2-D for both the acoustic wave equation [8] and Maxwell's equations [9] on second-order form. However, the efficiency of DG applied to systems of second-order hyperbolic equations on large 3-D applications is an open question.

In this paper we focus on: (1) deriving strictly stable HOFDM for the acoustic wave equation in discontinuous media, by combining second-derivative SBP operators and the simultaneous approximation term (SAT) method [5], and (2) introducing the technique in complex geometries by making use of the discrete Laplacian operator used in CDP[1] (an unstructured finite volume flow solver developed as part of Stanford's DOE-funded ASC Alliance program to perform LES in complex geometries). This approach is somewhat related to the DG method since they both make use of the penalty technique to handle the discontinuity in a truly non-overlap fashion.

The three reasons for introducing the SAT method instead of the recently developed projection method [24] to impose the discontinuous interface conditions are the following: (1) it is easier to implement (although, a detailed study is omitted here), (2) it is not limited to piecewise constant coefficients (see [24]), and (3) it is much more accurate (as will be shown in Section 4).

The two main reasons for introducing computational tools from CDP are the following: (1) it allows us to handle huge problems in complex geometries, and (2) it makes it easier to isolate and verify the accuracy and stability properties of the Laplacian operator used in CDP. (In spite of it's simplicity the second-order wave equation imposes a stricter stability requirement [24] on the discrete Laplacian operator than when used for parabolic problems like the Navier–Stokes equations).

In Section 2 we introduce some definitions and discuss the SBP property for the 1-D case, and show how to impose the boundary and interface conditions in discontinuous media using SAT. In Section 3 we will show how to implement this technique in complex geometries using the unstructured finite volume method. In Section 4 we will verify the accuracy and stability properties, by performing numerical computations in 1-D and 3-D. A direct comparison between the SAT method and the Projection method will be done for the 1-D case. In Section 5 we present our conclusions.

In this article, we only consider acoustic waves. The extension to handle for example elastic waves [15,2,36] with an analogous approach will be dealt with in a forthcoming paper.

2. The finite difference method

For clarity we will restrict the analysis to 1-D in this section. The extension to 2-D and 3-D (see for example [24,26,25]) is straightforward using 1-D SBP finite-difference operators.

We begin with a short description and some definitions (for more details, see [20,31,23]). Let the inner product for real-valued functions $u, v \in L^2[-1,1]$ be defined by $(u,v) = \int_{-1}^{1} u\, v\, w\, \mathrm{d}x$, $w(x) > 0$, and let the corresponding norm be $\|u\|_w^2 = (u,u)$. The domain $(-1 \leqslant x \leqslant 1)$ is discretized using $2N+1$ equidistant grid points

$$x_i = i\, h, \quad i = 0, 1 \ldots, 2N, \quad h = \frac{2}{N}.$$

[1] CDP is named after Charles David Pierce (1969–2002).