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Short Note

On the dissipation mechanism of upwind-schemes in the low Mach number regime: A comparison between Roe and HLL

Felix Rieper

Institut für Atmosphäre und Umwelt, Goethe-Universität Frankfurt, Altenhöferallee 1, D-60438 Frankfurt am Main, Germany

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ABSTRACT

It is well known that standard upwind schemes for the Euler equations face a number of problems in the low Mach number regime: stiffness, cancellation and accuracy problems. A new aspect of the *accuracy problem*, presented in this paper, is the dependence on the type of flux solver: while the accuracy of the HLL scheme massively decreases for $Ma \to 0$ on a given triangular mesh, the Roe scheme remains accurate, i.e. flows of arbitrarily small Mach numbers can – at least in principle – be simulated on a fixed triangular mesh. We give an asymptotic analysis of this phenomenon and present a number of numerical results.

1. Introduction

Schemes, originally designed to calculate compressible flow, encounter three problems in low Mach number flow. Firstly, the speeds of acoustic waves and flow phenomena are of different orders of magnitude – their ratio is measured by the Mach number. Low Mach numbers slow down the calculation of phenomena on the time scale of the flow such as heat or water transport (*stiffness problem*). Secondly, the pressure variable has to accommodate a constant background pressure of order $\mathcal{O}(1)$ and the physically relevant pressure variations of order $\mathcal{O}(Ma^2)$, which leads to numerical round-off errors (*cancellation problem*). And thirdly, for stability reasons, upwind schemes introduce artificial viscosity, which depends on the Mach number. In certain settings this can cause the truncation error to be $\mathcal{O}(1/Ma)$, i.e. to grow with decreasing Mach numbers on a given mesh, and thus preventing the numerical solution to approximate inviscid, incompressible flow (*accuracy problem*).

The *cancellation problem* can be avoided by working only with the fluctuation quantities introduced in the wave propagation approach by Leveque [1]. This approach was applied to low Mach number flow by Sesterhenn et al. [2].

To overcome the *stiffness and accuracy problem in steady flow simulations* a variety of time-derivative or flux preconditioning techniques have been developed and applied to compressible (and incompressible) solvers for the inviscid flow

E-mail address: rieper@iau.uni-frankfurt.de

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equations, such as Turkel's approach, [3,4], or the characteristic time stepping approach by van Leer et al. [5]. The stiffness is reduced by (almost) equalising the propagation speeds of the different waves for $Ma \to 0$, which accelerates the convergence to steady state. At the same time, the artificial viscosity is tuned correctly for all characteristic waves and thus the accuracy problem is circumvented. Preconditioning in the context of viscous flow was dealt with by Choi and Merkle [6]. They report on the absence of the accuracy problem down to $Ma = 10^{-6}$ for their preconditioning methods, which are implemented on grids with quadrilateral cells.

The *accuracy problem for transient flow* was explicitly addressed by Guillard and Viozat [7]. Their asymptotic analysis of the Roe scheme explains the appearance of a pressure term of wrong order as $Ma \to 0$ on *Cartesian grids*. They propose a viscosity–matrix preconditioning to fix the problem.

It is worth mentioning that Discontinuous Galerkin (DG) schemes do not show the accuracy problem as much as finite volume schemes as shown in [8]. Nevertheless, Bassi et al. show that preconditioning improves accuracy and efficiency of DG schemes in the low Mach number regime [9].

Thornber et al. [10] show that there is a one-to-one relation between dissipation of kinetic energy and increase in entropy. Their analysis reveals an unphysical entropy production of Godunov-type schemes at low Mach numbers related to the jumps at the cell interfaces. The aim to minimise these jumps is achieved with an improved reconstruction method presented by Thornber et al. [11]. A fifth-order in space reconstruction is changed in a way so that the jump in the normal component of the velocity in the Riemann problem is reduced. The approach results in a MUSCL scheme capable of calculating low Mach number flows on Cartesian meshes.

Despite the advances attained in the past, simulating low Mach number flows remains a challenge and still has open questions, such as: how to overcome the stiffness while maintaining time accuracy and computing efficiency; and can the reconstruction fix for low Mach number flows presented in [11] be applied to unstructured grids and to second-order reconstructions widely used in practice?

An interesting property of first-order Godunov-type schemes on *triangular meshes* is the absence of these jumps in the normal velocity component. Therefore, these schemes do not show the accuracy problem at low Mach numbers on triangular finite volume cells. This effect is demonstrated and proved with an asymptotic analysis in two dimensions by Rieper and Bader [12], and generalised by Guillard [13].

The failure of certain flux solvers such as HLL even on triangular meshes initiated the analysis presented here. We compare the dissipation rate of the individual characteristic waves for two common flux solvers, Roe and HLL. In general, only a certain class of solvers, which resolve the contact waves explicitly, have Mach-number independent dissipation of the contact wave and are thus able to approximate low Mach number flow. In Section 2 a characteristic analysis is done for the Roe scheme. Section 3 is dedicated to the HLL scheme and a generalisation of the approach for arbitrary flux solvers is given in Section 4. Numerical results corroborating the analysis are given in Section 5.

1.1. Governing equations

The 2D-Euler equations in conservation form are

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x + \mathbf{g}(\mathbf{q})_y = 0,$$

with

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(\rho e + p) \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(\rho e + p) \end{bmatrix},$$

with density ρ, velocity $(u, v)^T$, total specific energy e and pressure p. To close the equations, thermodynamic relations are needed. We use the perfect gas law $p = (\gamma - 1)\rho\varepsilon$, with the adiabatic index γ and $\varepsilon = e - \frac{1}{2}(u^2 + v^2)$ as internal specific energy.

In this study we focus on the one-dimensional behaviour of numerical schemes and therefore assume only plane waves – parallel to the y-axis without loss of generality – so that the 2D-equations collapse to a one-dimensional problem given by

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = 0. \tag{1.1}$$

Note, that $\mathbf{q}$ and $\mathbf{f}$ still depend on *two* velocity components: the normal velocity u and the transverse velocity v, so that shear waves are possible. In the following we analyse the numerical approximation of this equation and refer to it as the *one-dimensional model case*.

System (1.1) can be (locally) transformed using the right eigenvectors $\mathbf{r}_i$ of the Jacobian $A = \mathrm{d}\mathbf{f}/\,\mathrm{d}\mathbf{q}$ with the transformation matrix $R = [\mathbf{r}_1, \ldots, \mathbf{r}_4]$ and its inverse $R^{-1} = L = [\mathbf{l}_1, \ldots, \mathbf{l}_4]$, to obtain the characteristic form of the Euler equations

$$\frac{\partial \mathbf{w}}{\partial t} + \Lambda \frac{\partial \mathbf{w}}{\partial x} = 0,$$

where the characteristic variables are given by