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Capacitance matrix technique for avoiding spurious eigenmodes in the solution of hydrodynamic stability problems by Chebyshev collocation method

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ABSTRACT

We present a simple technique for avoiding physically spurious eigenmodes that often occur in the solution of hydrodynamic stability problems by the Chebyshev collocation method. The method is demonstrated on the solution of the Orr–Sommerfeld equation for plane Poiseuille flow. Following the standard approach, the original fourth-order differential equation is factorised into two second-order equations using a vorticity-type auxiliary variable with unknown boundary values which are then eliminated by a capacitance matrix approach. However the elimination is constrained by the conservation of the structure of matrix eigenvalue problem, it can be done in two basically different ways. A straightforward application of the method results in a couple of physically spurious eigenvalues which are either huge or close to zero depending on the way the vorticity boundary conditions are eliminated. The zero eigenvalues can be shifted to any prescribed value and thus removed by a slight modification of the second approach.

1. Introduction

Spectral methods are known to achieve exponential convergence rate [3], which makes them particularly useful for solving numerically demanding differential eigenvalue problems which arise in hydrodynamic stability analysis [11]. Unfortunately, besides providing accurate and efficient solutions for a certain number of leading eigenvalues, spectral methods often produce physically spurious unstable modes, which cannot be removed by increasing the numerical resolution [7]. For detailed discussion of these modes we refer to Boyd [2]. Such physically spurious eigenvalues can appear in all types of spectral methods including Galerkin [15], tau [4] and collocation approximations [3], unless some kind of *ad hoc* approach is applied to avoid them. In the Galerkin method, spurious eigenvalues can be removed by using the basis functions also as the test functions instead of separate Chebyshev polynomials [16]. A number of approaches avoiding spurious eigenvalues have also been found for the tau method [6,10,9]. The same can be achieved also for the collocation (or pseudospectral) method by using two distinct interpolating polynomials [8]. Following the approach of McFadden et al. [10] for the tau method, Huang and Sloan [8] use a Lagrange interpolating polynomial for second-order terms which is by two orders lower than the Hermite interpolant used for other terms. The choice of the latter polynomial depends on the particular combination of the boundary conditions for the problem to be solved [14, p. 493].

The objective of this paper is to present a simple method avoiding spurious eigenmodes in the Chebyshev collocations method which uses only the Lagrange interpolating polynomial applicable to general boundary conditions. Our approach

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is based on the capacitance matrix technique which is used to eliminate fictitious boundary conditions for a vorticity-type auxiliary variable. The elimination can be performed in two basically different ways which respectively produce a pair of infinite and zero spurious eigenvalues. The latter can be shifted to any prescribed value by a simple modification of the second approach. The main advantage of our method is not only its simplicity but also applicability to more general problems with complicated boundary conditions.

The paper is organised as follows. In the next section we introduce the Orr–Sommerfeld problem for plane Poiseuille flow, which is a standard test case for this type of method. Section 3 presents the basics of the Chebyshev collocation method that we use. The elimination of the vorticity boundary conditions, which constitutes the basis of our method, is performed in Section 4. Section 5 contains numerical results for the Orr–Sommerfeld problem of plane Poiseuille flow. The paper is concluded by a summary of results in Section 6.

2. Hydrodynamic stability problem

The method will be developed by considering the standard hydrodynamic stability problem of plane Poiseuille flow of an incompressible liquid with density ρ and kinematic viscosity ν driven by a constant pressure gradient $\nabla p_0 = -\boldsymbol{e}_x P_0$ in the gap between two parallel walls located $z = \pm h$ in the Cartesian system of coordinates with the x and z axes directed streamwise and transverse to the walls, respectively. The velocity distribution $\boldsymbol{v}(\boldsymbol{r}, t)$ is governed by the Navier–Stokes equation

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\rho^{-1}\nabla p + \nu \nabla^2 \boldsymbol{v} \tag{1}$$

and subject to the incompressibility constraint $\nabla \cdot \boldsymbol{v} = 0$. Subsequently, all variables are non-dimensionalised by using h and h^2/ν as the length and time scales, respectively. Note that instead of the commonly used maximum flow velocity, we employ the viscous diffusion speed ν/h as the characteristic velocity. This non-standard choice will allow us to test our numerical method against the analytical eigenvalue solution for a quiescent liquid.

The problem above admits a rectilinear base flow $\boldsymbol{v}_0(z) = Re\bar{u}(z)\boldsymbol{e}_x$, where $\bar{u}(z) = 1 - z^2$ is the parabolic velocity profile and $Re = U_0 h/\nu$ is the Reynolds number defined in terms of the maximum flow velocity $U_0 = P_0 h^2/(2\rho\nu)$. Stability of this base flow is analysed with respect to small-amplitude perturbations $\boldsymbol{v}_1(\boldsymbol{r}, t)$ by searching the velocity as $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1$. Since the base flow is invariant in both t and $\boldsymbol{x} = (x, y)$, perturbation can be sought as a Fourier mode

$$\boldsymbol{v}_1(\boldsymbol{r}, t) = \hat{\boldsymbol{v}}(z)e^{\lambda t + i\boldsymbol{k}\cdot\boldsymbol{x}} + \text{c.c.}, \tag{2}$$

defined by a complex amplitude distribution $\hat{\boldsymbol{v}}(z)$, temporal growth rate λ and the wave vector $\boldsymbol{k} = (\alpha, \beta)$. The incompressiblity constraint, which takes the form $\boldsymbol{D} \cdot \hat{\boldsymbol{v}} = 0$, where $\boldsymbol{D} \equiv \boldsymbol{e}_z \frac{d}{dz} + i\boldsymbol{k}$ is a spectral counterpart of the nabla operator, is satisfied by expressing the component of the velocity perturbation in the direction of the wave vector as $\hat{u}_\parallel = \boldsymbol{e}_\parallel \cdot \hat{\boldsymbol{v}} = ik^{-1}\hat{w}'$, where $\boldsymbol{e}_\parallel = \boldsymbol{k}/k$ and $k = |\boldsymbol{k}|$. Taking the *curl* of the linearised counterpart of Eq. (1) to eliminate the pressure gradient and then projecting it onto $\boldsymbol{e}_z \times \boldsymbol{e}_\parallel$, after some transformations we obtain the Orr–Sommerfeld equation

$$\lambda \boldsymbol{D}^2 \hat{w} = \boldsymbol{D}^4 \hat{w} + i\alpha Re(\bar{u}'' - \bar{u}\boldsymbol{D}^2)\hat{w}, \tag{3}$$

which is written in a non-standard form corresponding to our choice of the characteristic velocity. Note that the Reynolds number appears in this equation as a factor at the convective term rather than its reciprocal at the viscous term as in the standard form. As a result, the growth rate λ differs by a factor Re from its standard definition. The same difference, in principle, applies also to the velocity perturbation amplitude which, however, is not important as long as only the linear stability is concerned. In this form, Eq. (3) admits a regular analytical solution at $Re = 0$, which is used as a benchmark for the numerical solution in Section 5.

The no-slip and impermeability boundary conditions require

$$\hat{w} = \hat{w}' = 0 \quad \text{at} \quad z = \pm 1. \tag{4}$$

Because three control parameters Re and (α, β) appear in Eq. (3) as only two combinations αRe and $\alpha^2 + \beta^2$, solutions for oblique modes with $\beta \neq 0$ are equivalent to the transverse ones with $\beta = 0$ and a larger α and, thus, a smaller Re which keep both parameter combinations constant [5]. Therefore, it is sufficient to consider only the transverse perturbations $(k = \alpha)$.

The first step in avoiding spurious eigenvalues in the discretizied version of Eq. (3) to be derived in the following section is to represent Eq. (3) as a system of two second-order equations [7]

$$\lambda\hat{\zeta} = \boldsymbol{D}^2\hat{\zeta} + i\alpha Re(\bar{u}''\hat{w} - \bar{u}\hat{\zeta}), \tag{5}$$

$$\hat{\zeta} = \boldsymbol{D}^2\hat{w}, \tag{6}$$

where $\hat{\zeta}$ is a vorticity-type auxiliary variable which has no explicit boundary conditions.

3. Chebyshev collocation method

The problem is solved numerically using a collocation method with $N + 1$ Chebyshev–Gauss–Lobatto nodes

$$z_i = \cos(i\pi/N), \quad i = 0, \cdots, N. \tag{7}$$