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Positivity-preserving DG and central DG methods for ideal MHD equations ☆

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A B S T R A C T

Ideal MHD equations arise in many applications such as astrophysical plasmas and space physics, and they consist of a system of nonlinear hyperbolic conservation laws. The exact density ρ and pressure p should be non-negative. Numerically, such positivity property is not always satisfied by approximated solutions. One can encounter this when simulating problems with low density, high Mach number, or much large magnetic energy compared with internal energy. When this occurs, numerical instability may develop and the simulation can break down. In this paper, we propose positivity-preserving discontinuous Galerkin and central discontinuous Galerkin methods for solving ideal MHD equations by following [X. Zhang, C.-W. Shu, Journal of Computational Physics 229 (2010) 8918–8934]. In one dimension, the positivity-preserving property is established for both methods under a reasonable assumption. The performance of the proposed methods, in terms of accuracy, stability and positivity-preserving property, is demonstrated through a set of one and two dimensional numerical experiments. The proposed methods formally can be of any order of accuracy.

1. Introduction

In this paper, we continue our investigation in developing highly accurate and robust numerical methods for ideal MHD equations [13–15,25]. This system models many important problems in a wide range of applications such as astrophysical plasmas and space physics, and it consists of a set of nonlinear hyperbolic conservation laws,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1.1a}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \left[\rho \mathbf{u}\mathbf{u}^{\mathrm{T}} + \left(p + \frac{1}{2}|\mathbf{B}|^2 \right)\mathbf{I} - \mathbf{B}\mathbf{B}^{\mathrm{T}} \right] = 0, \tag{1.1b}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{B}^{\mathrm{T}} - \mathbf{B}\mathbf{u}^{\mathrm{T}}) = 0, \tag{1.1c}$$

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$$\frac{\partial E}{\partial t}+\nabla\cdot\left[\left(E+p+\frac{1}{2}|\mathbf{B}|^2\right)\mathbf{u}-\mathbf{B}(\mathbf{u}\cdot\mathbf{B})\right]=0, \tag{1.1d}$$

with an additional divergence-free constraint

$$\nabla\cdot\mathbf{B}=0. \tag{1.2}$$

Here ρ is the density, p is the hydrodynamic pressure, $\mathbf{u}=(u_x,u_y,u_z)^{\mathrm{T}}$ is the velocity field, and $\mathbf{B}=(B_x,B_y,B_z)^{\mathrm{T}}$ is the magnetic field. The total energy E is given by $E=\frac{1}{2}\rho|\mathbf{u}|^2+\frac{1}{2}|\mathbf{B}|^2+\frac{p}{\gamma-1}$ with γ as the ratio of the specific heats. We use the superscript T to denote the vector transpose. In addition, $\mathbf{I}$ is the identity matrix, $\nabla\cdot$ is the divergence operator, we further denote the momentum $\rho\mathbf{u}$ as $\mathbf{m}=(m_x,m_y,m_z)^{\mathrm{T}}$. Eqs. (1.1a), (1.1b), and (1.1d) are from the conservation of mass, momentum, and energy, and (1.1c) is the magnetic induction system. With compatible initial and boundary conditions, the divergence-free constraint (1.2) can be derived from the magnetic induction equations.

Besides the standard difficulty in simulating nonlinear hyperbolic equations, robust numerical algorithms for ideal MHD equations often require the divergence-free constraint in (1.2) being properly imposed [7,10,21,3]. In [13–15,25], we proposed and investigated strategies to obtain locally or globally divergence-free approximations for the magnetic field in discontinuous Galerkin (DG) and central DG framework. Besides the designed high order accuracy, these divergence-free methods demonstrate much improved numerical stability than the base methods without any divergence-free treatment. On the other hand, in the simulation of some examples including certain cloud–shock interaction problems, it is observed that the appearance of negative pressure can also lead to numerical instability, and such instability may not be removed by simply working with divergence-free schemes. In fact both density ρ and pressure p in exact solutions should be non-negative. Numerically, such positivity property is not always satisfied by approximated solutions, and this may cause the loss of hyperbolicity of the system and lead to instability of the simulation. For instance, one can encounter this when simulating problems with low density, high Mach number, or much large magnetic energy compared with internal energy. In this paper, we are interested in developing high order DG and central DG methods for (1.1), (1.2) which preserve positivity of both density and pressure. More specifically, we propose positivity-preserving limiters with which the cell average of the DG or central DG solution has positive density and pressure at discrete time t_n as long as they are initially positive. It is in general difficult to design positivity-preserving schemes which also satisfy the divergence-free constraint exactly. In this paper, the positivity-preserving limiters are presented for standard DG and central DG methods defined in Sections 3.1 and 3.2 where the divergence constraint is not considered. We want to point out that by utilizing the intrinsic local nature of both methods, one can also apply locally divergence-free approximations [13] straightforwardly in the present framework without affecting the positivity-preserving property of the overall algorithm (see also Remark 3.1).

In the context of ideal MHD equations, a positivity-preserving limiter was designed and analyzed in [22] for a second order finite volume method. The resulting scheme is conservative in one dimension, and it is nonconservative in higher dimensions with a source term added to the magnetic induction equation and properly discretized in order to take into account the normal jump in the magnetic field. Such modified magnetic induction equation allowing magnetic monopoles was also used in [12] for a positive scheme combining both HLL and Roe methods. In [5], a hybrid strategy was proposed for the positivity of pressure. It involves a linearized Riemann solver working directly with the entropy density equation instead of the total energy Eq. (1.1d) in the absence of magnetosonic shocks, and a standard Riemann solver based on (1.1) elsewhere. The overall strategy relies on switches to indicate where each Riemann solver should be applied.

On the other hand, for compressible Euler equations (which are the same as ideal MHD equations when the magnetic field is zero), an innovative positivity-preserving technique was recently introduced and analyzed by Zhang and Shu in [27] for finite volume methods and DG methods with arbitrary order of accuracy. This technique can be regarded as generalization of the maximum-principle-satisfying limiters for scalar conservation laws [26] and the positivity-preserving schemes for compressible Euler equations in [17]. It starts with a first order positivity-preserving scheme as a building block, followed by a necessary condition to ensure the positivity-preserving property of the methods of higher order accuracy. A simple local limiter is then designed and analyzed to enforce the sufficient condition without destroying the accuracy and conservation of the schemes. The limiter was first presented when the time discretization is forward Euler method, then high order accuracy in time is achieved with the use of strong stability preserving (SSP) time discretizations which can be written as a convex combination of forward Euler methods. In the present work, we apply the positivity-preserving technique of Zhang and Shu to DG methods in solving ideal MHD system, and we also propose such technique to central DG methods. In one dimension, the positivity-preserving property of both the DG and central DG methods is established theoretically under a reasonable assumption. In higher than one dimension, the numerically relevant Riemann problem allows nonzero divergence in the magnetic field, therefore the numerical divergence often needs to be taken into account in devising positivity-preserving schemes, see [22]. We here formally extend the proposed positivity-preserving limiters to two dimensions as in [27]. Though without rigorous analysis, the performance of the methods in terms of accuracy, stability, and being positivity-preserving, are successfully demonstrated through a set of one and two dimensional numerical experiments. Both DG and central DG methods use piecewise smooth functions as approximations, and they have proved themselves to be a good candidate to accurately and reliably simulate many linear and nonlinear problems including nonlinear conservations laws [8,16]. Compared with DG methods, central DG methods evolve two copies of numerical solutions and do not use any numerical flux (which is an approximate Riemann solver). This implies that one cannot rely on properly designed numerical Riemann