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A multilayer shallow water system for polydisperse sedimentation

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A B S T R A C T

This work considers the flow of a fluid containing one disperse substance consisting of small particles that belong to different species differing in size and density. The flow is modelled by combining a multilayer shallow water approach with a polydisperse sedimentation process. This technique allows one to keep information on the vertical distribution of the solid particles in the mixture, and thereby to model the segregation of the particle species from each other, and from the fluid, taking place in the vertical direction of the gravity body force only. This polydisperse sedimentation process is described by the well-known Masliyah–Lockett–Bassoon (MLB) velocity functions. The resulting multilayer sedimentation-flow model can be written as a hyperbolic system with nonconservative products. The definitions of the nonconservative products are related to the hydrostatic pressure and to the mass and momentum hydrodynamic transfer terms between the layers. For the numerical discretization a strategy of two steps is proposed, where the first one is also divided into two parts. In the first step, instead of approximating the complete model, we approximate a reduced model with a smaller number of unknowns. Then, taking advantage of the fact that the concentrations are passive scalars in the system, we approximate the concentrations of the different species by an upwind scheme related to the numerical flux of the total concentration. In the second step, the effect of the transference terms defined in terms of the MLB model is introduced. These transfer terms are approximated by using a numerical flux function used to discretize the 1D vertical polydisperse model, see Bürger et al. [R. Bürger, A. García, K.H. Karlsen, J.D. Towers, A family of numerical schemes for kinematic flows with discontinuous flux, J. Eng. Math. 60 (2008) 387–425]. Finally, some numerical examples are presented. Numerical results suggest that the multilayer shallow water model could be adequate in situations where the settling takes place from a suspension that undergoes horizontal movement.

1. Introduction

1.1. Scope

Numerous applications involve the flow of a mixture of one substance, for example solid mineral particles or oil droplets in an emulsion, dispersed in a continuous phase, say a liquid or gas. In many cases, the disperse substance consists of small

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particles that belong to different species differing in some characteristic quantity such as size or density. As a consequence, the polydisperse mixture does not move as one phase; rather, the different species segregate and create areas of different composition. In many applications, practitioners are most interested in this differential movement of the species, which is frequently described by spatially one-dimensional models. In most circumstances, the diameter of the particles is small, which justifies identifying each species with a continuous phase. The resulting models usually give rise to a strongly coupled system of nonlinear first-order conservation laws for the volume fractions of the solids species.

In many other applications, we are not only interested in this differential movement of the species but also in the fluid dynamics of the flow convecting the particulate suspensions, for instance the transport of soils, silt and sand in rivers and estuaries. In those cases, a full three-dimensional (3D) model could be considered, but the computational cost of solving such models is largely increased since in two or three space dimensions, not only a multi-dimensional version of the above-mentioned system of conservation laws, but also additional equations of motion (e.g., the Stokes or Navier–Stokes system) for the flow field of the mixture must be solved.

A common approach to model suspended sediment transport in shallow regimes is to use a Saint–Venant or shallow water model combined with passive transport equations for the different species. These models are obtained by averaging the original 3D equations along the height of the fluid and allow one to simulate sediment transport with a relative small computational cost (see, e.g. [16,21,22,28]). The drawback of these models is that they only take into account the mean depth-average concentration of solid particles in suspension. Thus, the vertical distribution and settling of the particles suspended within the fluid is not described.

The objective of this paper is to derive and implement a computational model for polydisperse sedimentation that takes into account the differential movement of the species as well as the dynamics of the flow. This will be achieved by a multilayer Saint–Venant approach (see for example [1,3,4,6,33]).

As is shown in [2,4], numerical simulations by using a multilayer approach allow one to recover interesting properties that are not observed when using just a hydrostatic shallow water model. Moreover, numerical results obtained with the multilayer approach show a good agreement with those obtained for a 3D free-surface Navier–Stokes system when the hydrostatic assumption is in effect. In [2,4] it is shown that the multilayer approach provides an alternative to the solution of the free-surface Navier–Stokes system, leading to a precise description of the vertical profile of the horizontal velocity while preserving the robustness and the computational efficiency of the usual Saint–Venant system. The multilayer technique has also been used for density-stratified flows in [5] where similar conclusions have been obtained. We employ this technique for the simulation of free-surface fluids with polydisperse sedimentation. This technique allows us to keep information on the vertical distribution of the mixture.

On the other hand, it is difficult to compare exactly the computational cost of different numerical techniques, for example it depends on the efficiency of the coding process. But an interesting property of the multilayer approach is that it enables one to approximate free surface flows without an extra difficulty. Then, the multilayer model can be computationally less expensive than some other numerical techniques to approach the full 3D model for free-surface flows. For instance, in [4] the authors remark that a comparable test using 6 layers (1452 nodes, 2620 triangles for the 2D mesh) takes a CPU time of 10 minutes for the multilayer and 33 minutes for the hydrostatic Navier–Stokes solver.

1.2. Related work

Mathematical models for the one-dimensional sedimentation of polydisperse suspensions are important to many applications in chemical engineering, mineral processing, wastewater treatment, medicine, geology, and other areas; see, for example, [17,23,32,34,36,37] for applications and [8,10] for mathematical treatments. On the other hand, experimental and theoretical analyses of two- or three-dimensional scenarios, where the convective sediment transport is important, include [9,14,18,24,25].

Multilayer Saint–Venant models have been used to study flows with large friction coefficients, with significant water depth and/or with important wind effects, among others (see for example [3,6,33]). In these cases, the standard shallow water system is considered invalid since the horizontal velocity can hardly be approximated by a vertically constant velocity in the whole domain.

The multilayer approach consists in subdividing, in the vertical direction, the domain into layers. This way, the multilayer Saint–Venant system derived in [1] consists in a set of coupled Saint–Venant systems for each layer. It is noteworthy that the layers are assumed here to be advected by the flow. Then, it is considered that no mass exchange occurs between neighboring layers making the model physically close to non-miscible fluids simulation. It is also extended to 3D computations of free surface flows with friction and viscosity effects in [4].

A different multilayer model using a formal asymptotic analysis of the two-dimensional (2D) incompressible Navier–Stokes equations with a hydrostatic framework is proposed by Audusse et al. [6]. Each layer is described by its height and by a vertically constant horizontal velocity. The main improvement is that mass and momentum exchange between the layers are allowed. In order to close the system, the height of the layer is related to the total height of the fluid. Then, the unknowns of the system are the total height of the fluid and a constant horizontal velocity at each layer. The vertical velocity can be computed by postprocessing, taking into account the incompressibility of the fluid.

In [5] Audusse et al. present the extension of this model for free surface density-stratified flows. It approximates formally the Navier–Stokes equations with variable density, when it varies depending on a quantity such as temperature or salinity.