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Short Note

A limiter for PPM that preserves accuracy at smooth extrema

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Abstract

We present a new limiter for the PPM method of Colella and Woodward [P. Colella, P.R. Woodward, The Piecewise Parabolic Method (PPM) for gas-dynamical simulations, Journal of Computational Physics 54 (1984) 174–201] that preserves accuracy at smooth extrema. It is based on constraining the interpolated values at extrema (and only at extrema) using non-linear combinations of various difference approximations of the second derivatives. Otherwise, we use a standard geometric limiter to preserve monotonicity away from extrema. This leads to a method that has the same accuracy for smooth initial data as the underlying PPM method without limiting, while providing sharp, non-oscillatory representations of discontinuities.

Keywords: Upwind methods; PPM; Limiters

1. Introduction

One of the great successes in numerical methods for hyperbolic conservation laws has been the use of non-linear hybridization techniques, known as limiters, to maintain positivity and monotonicity in the presence of discontinuities and underresolved gradients. As originally formulated [3,14,5], these methods have the property that the truncation error is first-order accurate at all extrema, regardless of the accuracy of the underlying high-order method. This problem has been known since these methods were first introduced, and there have been a variety of methods proposed to deal with it. Typically, these have been based on the idea allowing the representation of solution values outside the range defined by the cell averages [16], while still suppressing oscillations at discontinuities and underresolved gradients. In particular, the methods proposed to solve the problem to obtain uniform high-order accuracy for smooth solutions [6,8,7,13,2,10] typically have used quite elaborate analytic and/or geometric constructions. In this note, we propose a particularly simple approach to solving this problem for the PPM method [4]. It is based on changing the PPM limiter at extrema (and only at extrema) using non-linear combinations of various difference approximations of the second derivatives. This

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leads to a method that has the same accuracy for smooth initial data as the underlying PPM method without limiting, while providing sharp, non-oscillatory representations of discontinuities.

2. Scalar advection

We will consider the linear advection equation in one space dimension

$$\frac{\partial a}{\partial t} + u\frac{\partial a}{\partial x} = 0 \tag{1}$$

We assume that we know at time step n the averages of a over finite-volume cells of length h

$$\langle a\rangle_j^n \approx \frac{1}{h}\int_{(j-1/2)h}^{(j+1/2)h} a(x, n\Delta t)\mathrm{d}x \tag{2}$$

The PPM method in [4] for computing $\langle a\rangle_j^{n+1}$ is a conservative finite difference method

$$\langle a\rangle_j^{n+1} = \langle a\rangle_j^n + \frac{u\Delta t}{h}\left(a_{j-\frac{1}{2}}^{n+\frac{1}{2}} - a_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right) \tag{3}$$

where $a_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ is the average of a parabolic interpolant over the interval swept out by the characteristics crossing the cell face at $(j+\frac{1}{2})h$

$$\begin{aligned} a_{j+\frac{1}{2}}^{n+\frac{1}{2}} &= \mathscr{I}_{j,+}(\sigma) \quad \text{if } u > 0 \\ &= \mathscr{I}_{j+1,-}(\sigma) \quad \text{otherwise} \end{aligned} \tag{4}$$

where $\sigma = |u|\Delta t/h$, and

$$\mathscr{I}_{j,+}(\sigma) \equiv \frac{1}{\sigma h}\int_{(j+\frac{1}{2})h-\sigma h}^{(j+\frac{1}{2})h} a_j^I(x)\mathrm{d}x \tag{5}$$

$$\mathscr{I}_{j,+}(\sigma) \equiv \frac{1}{\sigma h}\int_{(j-\frac{1}{2})h}^{(j-\frac{1}{2})h+\sigma h} a_j^I(x)\mathrm{d}x \tag{6}$$

The parabolic interpolant $a_j^I(x), x \in [(j-1/2)h, (j+1/2)h]$ is uniquely determined by the cell average $\langle a\rangle_j^n$ and the left and right extrapolated edge values $a_{j,\pm} = a_j^I(j \pm 1/2)h$.

$$a_j^I(x) = a_{j,-} + \xi(a_{j,+} - a_{j,-} + a_{6,j}(1-\xi)), \quad a_{6,j} = 6\langle a\rangle_j^n - 3(a_{j,-} + a_{j,+}) \tag{7}$$

$$\xi = \frac{x - jh}{h}, \quad 0 \leqslant \xi \leqslant 1 \tag{8}$$

For this choice of interpolant, the averages (5) and (6) are given by the following formulas:

$$\mathscr{I}_{j,+}(\sigma) = a_{j,+} - \frac{\sigma}{2}\left(a_{j,+} - a_{j,-} - (1 - \frac{2}{3}\sigma)a_{6,j}\right) \tag{9}$$

$$\mathscr{I}_{j,-}(\sigma) = a_{j,-} + \frac{\sigma}{2}\left(a_{j,+} - a_{j,-} + (1 - \frac{2}{3}\sigma)a_{6,j}\right) \tag{10}$$

It is easy to check that

$$\sigma\mathscr{I}_{j,+}(\sigma) + (1-\sigma)\mathscr{I}_{j,-}(1-\sigma) = \langle a\rangle_j^n, \quad 0 \leqslant \sigma \leqslant 1 \tag{11}$$

To complete the description of the algorithm, we must specify how the parabolic interpolant is computed, or, equivalently, how the $a_{j,\pm}$ are computed. In [4], this was done in two steps.

2.1. Interpolating face values

We compute high-order accurate approximations to a at cell edges

$$a_{j+\frac{1}{2}}^n = a\left((j+\frac{1}{2})\Delta x, n\Delta t\right) + \mathrm{O}(h^p), \quad p \geqslant 3 \tag{12}$$